

Matrix Exponentials and Linear Systems

Math 5250 — Week 4

The Matrix Exponential

Definition. For a real or complex square matrix A , let

$$\exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

We can make sense of this using the Jordan normal form.

Reducing to Jordan Form

There is an invertible matrix P such that $B = PAP^{-1}$ is in Jordan form. Since $PA^kP^{-1} = B^k$,

$$P \exp(A) P^{-1} = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots = \exp(B).$$

So we can define $\exp(A)$ if we can make sense of $\exp(B)$.

The exponential respects the block structure of B , so it suffices to compute the powers of a single Jordan block.

Powers of a Jordan Block

Write a $k \times k$ Jordan block as

$$J_k(\lambda) = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ & & & 1 & \lambda \end{pmatrix} = \lambda I + N, \quad N^k = 0.$$

Since N commutes with λI ,

$$(\lambda I + N)^m = \sum_{j=0}^m \binom{m}{j} \lambda^{m-j} N^j.$$

The power N^j is obtained from N by shifting the ones “down and to the left”: it has ones on the j^{th} subdiagonal.

The Exponential of a Jordan Block

It follows that

$$e^{J_k(\lambda)} = e^\lambda \left(I + \sum_{j=1}^{k-1} \frac{N^j}{j!} \right),$$

and similarly

$$e^{tJ_k(\lambda)} = e^{\lambda t} \left(I + \sum_{j=1}^{k-1} \frac{t^j N^j}{j!} \right).$$

$$B = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$I + B + \frac{B^2}{2!} + \dots = \frac{2\lambda N}{N^2}$$

$$B^2 = \begin{pmatrix} \lambda^2 & & \\ & \lambda^2 & \\ & & \lambda^2 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} \lambda^3 & & \\ & \lambda^3 & \\ & & \lambda^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3\lambda^2 & 0 & 0 \\ 0 & 3\lambda^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3\lambda & 0 & 0 \end{pmatrix}$$

$3\lambda^2 N \quad + 3\lambda N^2$

$$B^4 = \begin{pmatrix} \lambda^4 & & \\ & \lambda^4 & \\ & & \lambda^4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4\lambda^3 & 0 & 0 \\ 0 & 4\lambda^3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6\lambda^2 & 0 & 0 \end{pmatrix}$$

$$\exp(B) = \begin{pmatrix} e^\lambda & & \\ & e^\lambda & \\ & & e^\lambda \end{pmatrix} + e^{\lambda} I$$

$$\begin{pmatrix} 0 & 0 & 0 \\ e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 \end{pmatrix} = e^\lambda N$$

$$\frac{1}{2!} e^\lambda N^2$$

$$e^\lambda (I + N + \frac{N^2}{2!} + \dots)$$

First-Order Linear Systems

Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where the x_i are functions of t and $\dot{x}_i = dx_i/dt$.

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \exp(tA) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where the a_i are the initial conditions (the values of the x_i at $t = 0$).

$$\begin{aligned} \ddot{x} + a_1 \dot{x} + a_2 x + a_3 x &= 0 \\ \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} \\ &= A \exp(tA) \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} \end{aligned}$$

Solutions: Diagonal and Jordan Cases

If A is diagonal with entries $\lambda_1, \dots, \lambda_n$, the system breaks up into one-dimensional pieces with solutions

$$x_i = a_i e^{\lambda_i t}.$$

$$\exp(tA)$$

If $A = J_n(\lambda)$ is a Jordan block, then

$$x_1 = a_1 e^{\lambda t}$$

$$x_2 = e^{\lambda t} (a_2 + t a_1)$$

$$x_3 = e^{\lambda t} \left(a_3 + t a_2 + \frac{t^2}{2!} a_1 \right)$$

$$\vdots$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$X^2 + 1 = 0$$

$$\pm i$$

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$e^{ix}$$

$$\cos x + i \sin x$$

$$e^{-ix}$$

$$\cos x - i \sin x$$

$$e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$