Eigenvalues and Eigenvectors

Math 5250 — Week 4

Diagonalizability

Definition. A linear map $L : V \to V$ is **diagonalizable** if there is a basis E for V such that $[L]_E^E$ is a diagonal matrix. A matrix A is diagonalizable if $L_A : F^n \to F^n$ is diagonalizable.

A is diagonalizable if and only if it is similar to a diagonal matrix, i.e. PAP^{-1} is diagonal for some invertible P.

$L \rightsquigarrow [L]_E$

$\updownarrow$

$[L]_{E'}$

$[L]_{E'} = P[L]_E P^{-1}$

The characteristic polynomial

$\{x_1, \ldots, x_n\} = E$

$Ax_i = \lambda_i x_i$

$[A]_E = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ 0 & 0 & \ddots & \\ 0 & & 0 & \ddots \end{pmatrix}$

$A\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} = a_{11}\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}$

Definition. A nonzero vector x with $Ax = \lambda x$ is an **eigenvector** for A with **eigenvalue** λ; similarly for a linear map L.

$\underbrace{(A - \lambda I)}\, x = 0$

A (resp. L) is diagonalizable if and only if there is a basis of eigenvectors for A (resp. L).

Proposition. λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$. Any nonzero vector in the null space of $\lambda I - A$ is then an eigenvector for A with eigenvalue λ.

The characteristic polynomial (cont.)

$\det(P(xI - A)P^{-1})$
$= \det(xI - A)$

Definition. The **characteristic polynomial** of A is $P_A(x) = \det(xI - A)$ (similarly $P_L(x)$ for a linear map L).

Proposition. $P_A(x)$ is a monic polynomial in x of degree n.

A scalar λ is a root of $P_A(x)$ if and only if λ is an eigenvalue of A.

Definition. A field F is **algebraically closed** if every non-constant polynomial over F has a root in F.

$(xI - A)$

$$\det \begin{pmatrix} x - a_{11} & a_{12} & \cdots \\ a_{21} & x - a_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix} = x^n + \underbrace{\cdots}_{\text{lower order terms}}$$

The characteristic polynomial (cont.)

Proposition. If F is algebraically closed, then every linear map/matrix over F has an eigenvalue (and an associated eigenvector).

Proposition. The complex numbers are algebraically closed. (This is the Fundamental Theorem of Algebra.)

Corollary. Every complex square matrix has an eigenvector.

Example. This is false for real matrices: the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is $x^2 + 1$, which has no real roots.

The characteristic polynomial (cont.)

Proposition. Not every matrix is diagonalizable.

Example. Let $V = \mathbf{C}^2$ and

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The characteristic polynomial $P_A(x) = (x-1)^2$, so 1 is the only eigenvalue. The eigenspace $\{x : Ax = x\}$ is one-dimensional, spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so A has no basis of eigenvectors.

Proposition. If $L : V \to V$ has $n = \dim V$ distinct eigenvalues, then L is diagonalizable.

Eigenvectors for different eigenvalues are linearly independent

Corollary. If the characteristic polynomial $P_L(x)$ has n distinct roots, then L is diagonalizable.

Remark. The converse is false: the identity matrix has characteristic polynomial $(x-1)^n$ but is diagonal.

$x_1, \ldots, x_n$ eigenvectors

λ_i eigenvalues

$\lambda_i \neq \lambda_j$

suppose $\sum a_i x_i = 0$ minimal length
no smaller collection dependent

$A(\sum a_i x_i) = \sum a_i \lambda_i x_i = 0$

$a_i \neq 0$

$\lambda_1(a_1 x_1 + \ldots + a_n x_n) = 0$

$\lambda_1 a_1 x_1 + \lambda_2 a_2 x_2 + \lambda_n a_n x_n = 0$

$(\lambda_1 - \lambda_2) a_2 x_2$

$+ \ldots + (\lambda_1 - \lambda_n) a_n x_n = 0$

The Cayley-Hamilton Theorem

$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$(x-1)^2$

x^2+1 $\quad A^2 + I = 0$ $\quad A - 1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

For a polynomial $P(x) \in F[x]$, $P(A)$ (resp. $P(L)$) makes sense as a matrix (resp. linear map), and $[P(L)]_E^E = P([L]_E^E)$ for any basis E.

Theorem (Cayley-Hamilton). Let $P_A(x) = \det(xI - A)$ be the characteristic polynomial of A. Then $P_A(A) = 0$.

$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$

$a_i \in F$ (constants)

$P(A)$ where A is a matrix

$A^n + a_{n-1}A^{n-1} + \cdots + a_0 I$

Proof of Cayley-Hamilton

$$X^{adj} = \begin{pmatrix} p_{11}(x) & \cdots & \cdots \\ \vdots & & \\ \vdots & - & - & p_{nn}(x) \end{pmatrix}$$

Let $X = xI - A$. Then

$$XX^{adj} = \det(X)I = P_A(x)I.$$

Each entry of X^{adj} is a determinant of an $(n-1) \times (n-1)$ submatrix of X, hence a polynomial in x of degree at most $n-1$, so

$$X^{adj} = B_0 + xB_1 + \cdots + x^{n-1}B_{n-1}$$

for $n \times n$ matrices B_i.

Proof of Cayley-Hamilton (cont.)

$$XX^{adj} = P(x)\, I$$

Then $(xI - A)X^{adj} = P(x)\, I$

$$xIX^{adj} = \underline{xB_0 + \cdots + x^n B_{n-1}}, \qquad \underline{AX^{adj} = AB_0 + \cdots + x^{n-1} AB_{n-1}}.$$

Since $\underline{XX}^{adj} = x\underline{IX}^{adj} - \underline{AX}^{adj}$, $\quad X = xI - A$

$$P_A(x)I = -AB_0 + x(B_0 - AB_1) + \cdots + x^{n-1}(B_{n-2} - AB_{n-1}) + x^n B_{n-1}.$$

Substituting $x = A$ collapses the right side to a telescoping sum that vanishes.

Cayley-Hamilton and Invariant Subspaces

Proposition. If V is finite dimensional, $L : V \to V$ is linear, and $W \subset V$ is an L-invariant subspace, then

$$P_L(x) = P_{L|W}(x)\, P_{\overline{L}}(x),$$

where $P_{L|W}(x)$ is the characteristic polynomial of $L : W \to W$ and $P_{\overline{L}}(x)$ is the characteristic polynomial of the induced map $\overline{L} : V/W \to V/W$.

Definition. For a bounded complex $V^{\cdot}$ of finite dimensional vector spaces and a linear map $L : V^{\cdot} \to V^{\cdot}$, the characteristic polynomial of the complex is

$$P_{L|V^{\cdot}}(x) = \prod_i P_{L|V^i}(x)^{(-1)^i}.$$

This can be computed from the cohomology of the complex:

$$P_{L|V^{\cdot}}(x) = \prod_i P_{\overline{L}|H^i(V^{\cdot})}(x)^{(-1)^i}.$$