Trace and Determinant

Math 5250 — Week 2

Trace

Definition. For an $n \times n$ matrix A over a field F, the **trace** is

$$\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

Proposition. If A, B are $n \times n$ matrices over F, then $\mathrm{trace}(AB) = \mathrm{trace}(BA)$.

$tr(A) = \sum a_{ii}$ $\quad tr(B) = \sum b_{ii}$

$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$

$(AB)_{ii} = \sum_{r=1}^{n} a_{ir} b_{ri}$

$tr(AB) = \sum_{i=1}^{n} \sum_{r=1}^{n} a_{ir} b_{ri}$

$b_{ri} a_{ir}$

$tr(BA)$

Trace of a Linear Map

$$(BA)_{ii} = \sum_{r=1}^{n} b_{ir} a_{ri}$$

$$tr(BA) = \sum_i \sum_r b_{ir} a_{ri}$$

Let $f : V \to V$ be linear, V finite dimensional.

Definition. Choosing a basis E for V, $\mathrm{trace}(f) = \mathrm{trace}([f]_E^E)$.

This is independent of the choice of basis: for a different basis E',

$$\mathrm{trace}([f]_{E'}^{E'}) = \mathrm{trace}([f]_E^E).$$

$$[f]_{E'}^{E'} = P[f]_E^E P^{-1}$$

$$tr(PXP^{-1}) = tr((PX)P^{-1}) = tr(P^{-1}PX) = tr(X)$$

Trace, Subspaces, and Quotient Spaces

Let $L : V \to V$ be linear and $W \subset V$ an **invariant subspace** $(L(W) \subset W)$. Then L induces a well-defined map $\overline{L} : V/W \to V/W$ by $\overline{L}([v]) = [L(v)]$.

Proposition. With L_W the restriction of L to W,

$$\operatorname{trace}(L) = \operatorname{trace}(L_W) + \operatorname{trace}(\overline{L}).$$

$f(w) \subseteq W$

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d, "L_W"] & V \arrow[r] \arrow[d, "L"] & V/W \arrow[r] \arrow[d, "\overline{L}"] & 0 \\
0 \arrow[r] & W \arrow[r] & V \arrow[r] & V/W \arrow[r] & 0
\end{tikzcd}$$

Pick $w_1, \ldots, w_k$ a basis for W

Extend by $v_{k+1}, \ldots, v_n$ to get a basis for W. call this E

$[L]^E_E = \begin{pmatrix} a_1 & \\ a_2 & \\ 0 & \end{pmatrix}$ $\quad L(w_1) = a_1 w_1 + a_2 w_2 + \cdots a_k w_k$

$w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$

$$[L] = \left(\begin{array}{c|c} \begin{matrix} k \times k \\ A \end{matrix} & * \\ \hline 0 & \begin{matrix} B \\ n-k \times n-k \end{matrix} \end{array}\right)$$

$tr(L) = tr(A) + tr(B)$

$tr(A) = tr(L_w)$

$tr(B) = tr(\overline{L})$

Note that $[v_{k+1}], \ldots, [v_n]$ in V/W they are a basis for V/W.

$[a_1 \underline{w_1 + a_2 w_2 + \ldots + a_k w_k} + a_{k+1} v_{k+1} + \ldots + a_n v_n]$

$= [a_{k+1} v_{k+1} + \ldots + a_n v_n]$

$= \sum_{j=k+1}^{n} a_j [v_j]$

$L \quad ($

Trace and Short Exact Sequences

If $0 \to A \to B \to C \to 0$ is a short exact sequence and $L = (L_1, L_2, L_3)$ is a compatible linear map $(A, B, C) \to (A, B, C)$, then

$$\mathrm{trace}(L_1) - \mathrm{trace}(L_2) + \mathrm{trace}(L_3) = 0.$$

More generally, for a bounded complex $X^{\cdot}$ of finite dimensional spaces and a linear map $L : X^{\cdot} \to X^{\cdot}$,

$$\mathrm{trace}(L|X^{\cdot}) := \sum_i (-1)^i \mathrm{trace}(L_i) = \sum_i (-1)^i \mathrm{trace}(\overline{L}_i),$$

where $\overline{L}_i$ are the induced maps on cohomology.

$$\begin{tikzcd} \arrow[r] & X^i \arrow[r, "d_i"] \arrow[d, "L^i"] & X^{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "L^{i+1}"] & X^{i+2} \arrow[r, "d_{i+2}"] \arrow[d, "L^{i+2}"] & \cdots \\ & {} & {} & {} & \end{tikzcd}$$

$$X^i \xrightarrow{d_i} X^{i+1} \xrightarrow{d_{i+1}} X^{i+2}$$

$\cancel{d_i d_{i+1} = }$ $\quad d_{i+1} d_i = 0 \qquad \bar{L} : H^i(X) \to H^i(X)$

$$d_i L^i = L^{i+1} d_i \qquad = Z^i / B^i$$

$$Z^i = \ker d_i$$

$$B^i = \operatorname{image}(d_{i-1})$$

$$0 \to Z^{i+1} \to X^{i+1} \xrightarrow{d_{i+1}} B^{i+2} \to 0$$

$$\operatorname{tr}(L \mid X^{i+1}) = \operatorname{tr}(L \mid B^{i+2}) + \operatorname{tr}(L \mid Z^{i+1})$$

$$\sum (-1)^i \operatorname{tr}(L \mid X^i) = \sum_i (-1)^i \operatorname{tr}(L \mid B^{i+1}) + \sum (-1)^i \operatorname{tr}(L \mid Z^i)$$

$$= \sum (-1)^{i+1} \operatorname{tr}(L \mid B^i) + \sum (-1)^i \operatorname{tr}(L \mid Z^i)$$

$$= \sum (-1)^i \Big[\underbrace{\operatorname{tr}(L \mid Z^i) - \operatorname{tr}(L \mid B^i)}_{\operatorname{tr}(\bar{L} \mid H^i)} \Big]$$

$$0 \to B^i \to Z^i \to H^i \to 0 \qquad H^i = \cancel{B^i / Z^i} \; Z^i / B^i$$

Determinant

Proposition. There is a unique function $\det : M_n(F) \to F$ such that:

1. det is multilinear in the rows of A.
2. det is alternating: it vanishes when two rows are repeated (so swapping two rows negates det).
3. $\det(I) = 1$.

$A \in M_n(F)$

$$A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \qquad r_i \in (a_{i1}, \dots a_{in})$$

$\det(r_1, \ldots \ldots, r_n)$

$\det(r+s, r_2, \ldots \ldots, r_n)$

$= \det(r_1, r_2, \ldots \ldots, r_n) + \det(s, r_2, \ldots, r_n)$

$\det(ar_1, r_2, \ldots \ldots, r_n) = a\det(r_1, \ldots, r_n)$

$\det(r_1, \ldots, r_i + s_i, \ldots \ldots, r_n)$

$\det(r_1 + s_1, r_2 + s_2)$

$= \det(r_1, r_2 + s_2) + \det(s_1, r_2 + s_2)$

$= \boxed{\det(r_1, r_2) + \det(r_1, s_2) + \det(s_1, r_2) + \det(s_1, s_2)}$

$\det(r_1, r_1, r_3, \ldots \ldots, r_n) = 0$

$\det(r+s, r+s, \ldots \ldots \ldots) = 0$

$\det(r, s \ldots) + \det(s, r \ldots) = 0$

suppose $\det(s, r) = -\det(r, s)$

in char zero

$\det(s, s) = -\det(s, s)$

$= 0$

Effect of Row Operations on the Determinant

Assuming det exists as above:

a. Swapping two rows of A: $\det(A) = -\det(A')$.
b. Scaling a row by a: $\det(A) = a\det(A')$.
c. Adding a multiple of one row to another: $\det(A) = \det(A')$.

$\det(r_1, r_2, \ldots, r_n)$

$\det(r_1 + a r_j, r_2, \ldots, r_n)$

$= \det(r_1, r_2, \ldots, r_n) + a \det(r_j, r_2, \ldots, r_n)$ $= 0$ if $i = j$

Existence of the Determinant

- 1×1: $\det(a) = a$. 2×2: $\det \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc$.
- Inductively, with A_i the submatrix deleting row 1 and column i,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(A_i).$$

This is the cofactor expansion along the first row (or any row or column).

$$A \quad \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$\det A = a_{11} \det(\underline{A}_1) - a_{12} \det(\underline{A}_2) + a_{13} \det(\underline{A}_3)$$

Determinant is Multiplicative

Proposition. $\det(AB) = \det(A)\det(B)$.

Corollary. A matrix is invertible if and only if its determinant is nonzero.

$$A = E_1 \cdots\cdots E_k \cdot J$$

J has a zero row

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det(A) = 0$$

$$\det(A) = \det(E_1(E_2 \cdots E_k))$$

$$= \det(E_1)\det(E_2 \cdots E_k)$$

Permutations and the Determinant

For the standard row vectors e_i, $\det(e_{i_1}, \ldots, e_{i_n}) = \pm 1$; the permutation $i_1, \ldots, i_n$ is **even** if this is $+1$, **odd** if -1 — equivalently, according to the parity of the number of exchanges needed to sort it.

Expansion.

$$\det(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n} \det(e_{i_1}, \ldots, e_{i_n}),$$

a sum of $n!$ terms (the Leibniz formula).

The Adjugate Matrix

Definition. The **adjugate** A^{adj} has entries $\boxed{b_{ij} = (-1)^{i+j} \det(A_{ji})}$, where A_{ji} deletes row j and column i of A.

Proposition. $AA^{adj} = A^{adj} A = \det(A)\, I$.

Remark. This gives an explicit formula for the inverse: $A^{-1} = \det(A)^{-1} A^{adj}$.

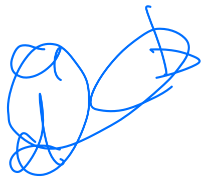
delete row 1, col 1

$b_{11} = (-1)^{1+1} \det(A_{11})$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ & \vdots & & \end{pmatrix} \qquad \begin{pmatrix} A_{11} & & \\ & & \\ & & \end{pmatrix}$$

$\cancel{b_{ij} = (-1)^{i+j} \det(A_{ij})}$

~~delete row i col j~~



big row 1, col 1

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

Cramer's Rule

For $Ax = b$ with $\det(A) \neq 0$,

$$x_j = \frac{\det(A_j(b))}{\det(A)},$$

$$x = \frac{1}{\det(A)} A^{adj} b$$

where $A_j(b)$ is A with column j replaced by b.

Remark. If A, b have integer entries and $\det(A) = \pm 1$, then x has integer entries.

Determinant of a Linear Map

Let $f : V \to V$ be linear, V finite dimensional over F.

Definition. $\det(f) = \det([f]_E^E)$ for any basis E.

This is well-defined: changing basis conjugates the matrix, which does not change the determinant.

Determinants and Short Exact Sequences

Proposition. If $0 \to A_1 \to A_2 \to A_3 \to 0$ is a short exact sequence of finite dimensional spaces and $L = (L_1, L_2, L_3)$ is a compatible linear map, then

$$\det(L_2) = \det(L_1)\det(L_3).$$

More generally, for a bounded complex $X^{\cdot}$ and $L : X^{\cdot} \to X^{\cdot}$,

$$\sum_i (-1)^i \det(L^i : X^i \to X^i) = \sum_i (-1)^i \det(\overline{L}^i : H^i(X^{\cdot}) \to H^i(X^{\cdot})).$$