Sums, Quotients, and Complexes

Math 5250 — Week 2

Direct Sums

Let V, W be vector spaces over a field $\mathbf{F}$. The **direct sum**

$$V \oplus W = \{(v, w) : v \in V,\ w \in W\}$$

is a vector space with $(v, w) + (v', w') = (v + v', w + w')$ and $a(v, w) = (av, aw)$.

- The direct sum of a finite family $\{V_i\}_{i=1}^{m}$ is the set of sequences $(v_1, \ldots, v_m)$, $v_i \in V_i$.
- For an index set I, $\oplus_{i \in I} V_i$ is the set of $f : I \to \bigcup_i V_i$ with $f(i) \in V_i$, nonzero for only finitely many i.
- $\mathbf{F}$ is a one-dimensional vector space over itself, and $\mathbf{F}^n = \oplus_{i=1}^{n} \mathbf{F}$.

Direct Sums: Dimension

V and W sit inside $V \oplus W$ as $(v, 0)$ and $(0, w)$, intersecting only in $(0, 0)$.

Proposition. If V, W are finite dimensional of dimensions n, m, then $V \oplus W$ is finite dimensional of dimension $n + m$; if $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ a basis of W, then

$$(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)$$

is a basis of $V \oplus W$.

$v = \sum a_i v_i \qquad w = \sum b_j w_j \qquad$ and $a_i = 0$, $b_j = 0$

$(v, w) = \sum a_i (v_i, 0) + \sum b_j (0, w_j) \qquad //$

$\sum a_i (v_i, 0) + \sum b_j (0, w_j) = 0 \qquad \left(\sum a_i v_i, \sum b_j w_j\right) = (0, 0)$

Subspace Sums

$\text{Span}(u_1, \ldots, u_k)$
$= \text{smallest subspace containing } u_1, \ldots, u_k$

$\langle v_i \rangle = 1 \text{ dim'l subspace spanned by } v_i$

$\langle u_1 \rangle + \langle u_2 \rangle + \cdots + \langle u_k \rangle$

Let U, W be subspaces of V. The **sum**

$$U + W = \{u + w : u \in U,\ w \in W\}$$

is a subspace of V.

- It is the smallest subspace of V containing both U and W.
- If $u_1, \ldots, u_n$ span U and $w_1, \ldots, w_k$ span W, then $u_1, \ldots, u_n, w_1, \ldots, w_k$ span $U + W$.
- $\dim(U + W) \leq \dim(U) + \dim(W)$, and this can be strict (e.g. $U = W$).

$\mathbb{R}^4$

U: $(1,0,0,0)$, $(0,1,0,0)$

W: $0,0,1,0$; $0,0,0,1$

Algorithm to Compute a Basis for $U + W$

$r + s = \dim(U) + \dim(W)$

$U, W \subseteq V \quad E = \{v_1, \dots, v_n\}$ basis for V $\quad n = \dim(V)$

$u_1, \dots, u_r$ basis for U $\quad [u_i]_E$ $\quad n \updownarrow \begin{bmatrix} [u_i]_E & [w_j]_E \end{bmatrix}$

$w_1, \dots, w_s$ $\quad [w_j]_E$

1. Choose a basis for V and spanning sets for U and W as above. Form the matrix A whose columns are the u_i and w_j in the chosen basis.
2. Row-reduce A to A'.
3. The columns of A corresponding to pivot columns of A' form a basis for $U + W$.
4. This basis is a subset of the union of the u_i and w_j.

$$\dim(U+W) = \operatorname{rk} \begin{bmatrix} [u_i]_E & [w_j]_E \end{bmatrix}$$

Internal Direct Sums

Suppose $x \in U+W$ is uniquely $u+w$ $u \in U, w \in W$

$f: U+W \to U \oplus W$ $\qquad$ $U \oplus W \to U+W$

$f(x) = (u, w)$ $\qquad$ $(u, w) \mapsto u+w$

Definition. If every $x \in U + W$ has a *unique* representation $x = u + w$ with $u \in U$, $w \in W$, then $U + W$ is an (internal) direct sum, isomorphic to $U \oplus W$.

Proposition. $U + W$ is a direct sum if and only if either:

a. $U \cap W = 0$, or

b. $\dim(U) + \dim(W) = \dim(U + W)$.

U

$\dim(U \oplus W) = \cancel{\dim(U)} + \dim$

$\dim(U+W)$

$$\boxed{\dim(U) + \dim(W) = \dim(U+W) + \dim(U \cap W)}$$

$U, W \subseteq V$

$U \oplus W \qquad U+W \subseteq V$

$f : (u, w) \longrightarrow u+w$

$f((u,w)+(u',w')) = f((u+u', w+w'))$

$= u+u'+w+w'$

$= f((u,w)) + f((u',w'))$

$\mathrm{null}(f) = ?$

$\{ (u,w) \mid f((u,w)) = 0 \}$

$u+w=0$

$u=-w \qquad u \in W \cap U$

$u \in U \cap W$

$(u,-u) \in \mathrm{null}(f)$

$U \cap W \subseteq \mathrm{null}(f)$

$\mathrm{null}(f) = U \cap W$

$\mathrm{image}(f) = U+W$

$\underline{U \cap W} \qquad U \oplus W \longrightarrow U+W$

$\dim(U \cap W) =$ nullity of f

$\dim(U+W) = \mathrm{rk}(f)$

$\dim(U \oplus W) = \dim(U+W) + \dim(U \cap W)$

A Dimension Formula

Proposition. For subspaces U, W of V,

$$\dim(U+W) = \dim(U \oplus W) - \dim(U \cap W) \cancel{= \dim(U) + \dim(W) - \dim}(U \cap$$

$$= \dim(U) + \dim(W) - \dim(U \cap W)$$

Equivalence Relations

Definition. An *equivalence relation* on X is a relation $x \sim y$ satisfying:

1. $x \sim x$ for all x.
2. $x \sim y \implies y \sim x$.
3. $x \sim y,\ y \sim z \implies x \sim z$.

An equivalence relation partitions X into equivalence classes $[x] = \{y \in X : y \sim x\}$; distinct classes are disjoint.

Fix $N > 0$ $\qquad a \equiv b \bmod N \iff N \mid a-b$

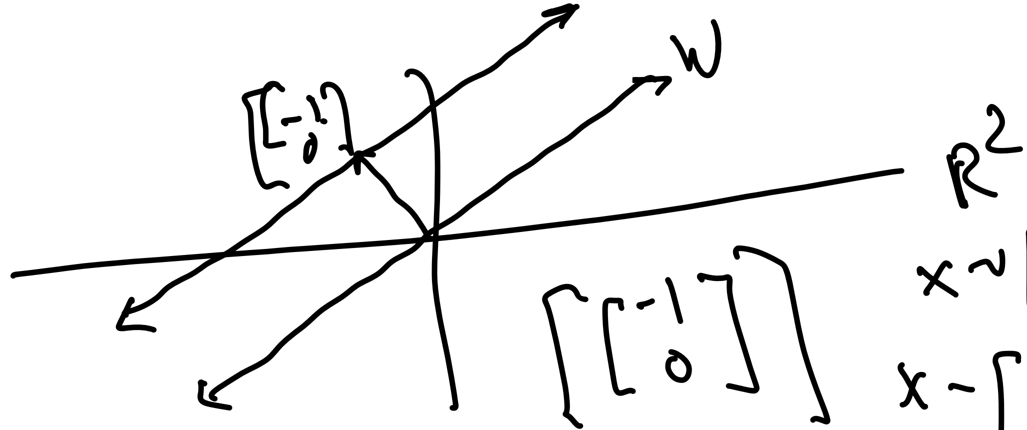
$[a]$ there are N of these $\qquad a = qN + r$

$[0], [1], \ldots\ldots [N-1]$

$a \sim r \qquad 0 \le r < N$

$N \mid a-r$

Quotient Spaces



$x \sim \begin{bmatrix}-1\\0\end{bmatrix}$

$x - \begin{bmatrix}-1\\0\end{bmatrix} = W$

Let V be a vector space over F and $W \subset V$ a subspace. Define $x \sim y \iff x - y \in W$.

$x = \begin{bmatrix}-1\\0\end{bmatrix} + W$

Definition. The set of equivalence classes $[x]$ is the **quotient space** V/W, a vector space over F with

$$\rightarrow \quad [x] + [y] = [x + y], \qquad a[x] = [ax].$$

$[x'] = [x] \qquad [y'] = [y]$

$x' = x + w_1$

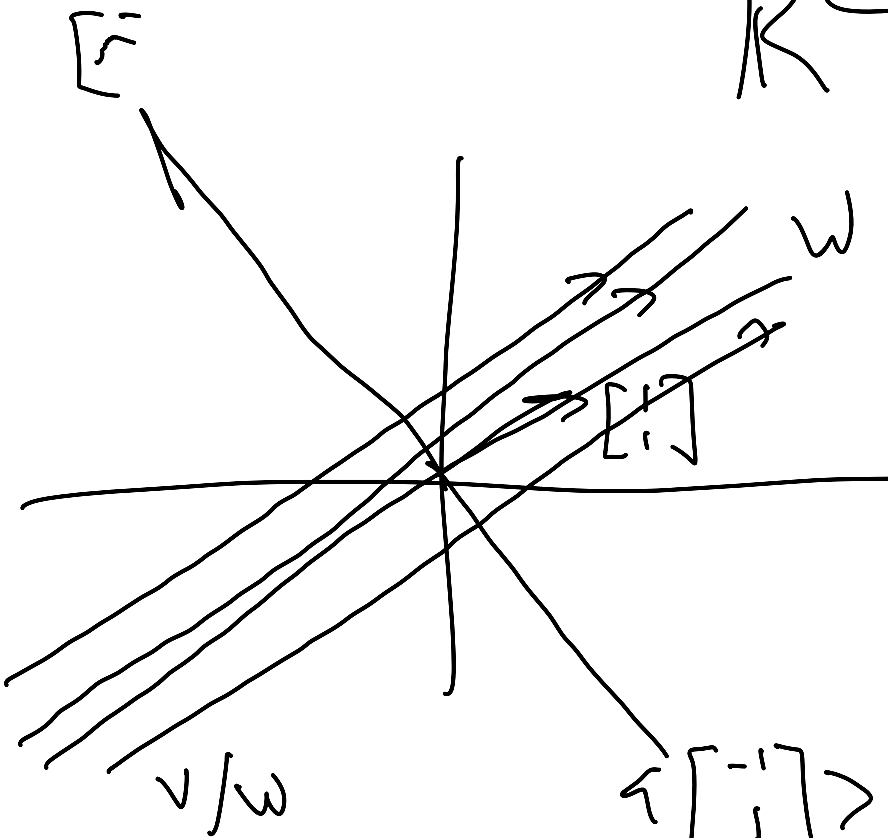
$y' = y + w_2$

$x' + y' = x + y + \overbrace{(w_1 + w_2)}^{\in W}$

$[x' + y'] = [x + y]$

$\mathbb{R}^2$

$W = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$



$$\left[\begin{bmatrix} a \\ b \end{bmatrix} \right]$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix}$$

$$\left\{ \left[\begin{bmatrix} -x \\ x \end{bmatrix} \right] \;\middle|\; x \in \mathbb{R} \right\}$$

$a = -x+y$
$b = x+y$

V/W is 1-dim'l

$\left[\begin{bmatrix} a \\ b \end{bmatrix} \right]$ is a basis as long as $a \neq b$.

Quotient Spaces: Dimension

Proposition. If V is finite dimensional and $W \subset V$, then

$$\dim(V/W) = \dim(V) - \dim(W).$$

(Via the linear map $f : V \to V/W$, $f(v) = [v]$, which is onto with $\mathrm{null}(f) = W$.)

$f(v_1+v_2) = [v_1+v_2] = [v_1]+[v_2] = f(v_1)+f(v_2)$

f surjective because every elt of V/W is $[v]$

$\mathrm{null}(f) = W$ $\quad f(v) = [v] = [0] \iff \underset{v \in W}{v - 0 \in W}$

Universal Property of Quotients

Proposition. Let $f : V \to W$ be linear and $H \subset V$ contained in $\ker(f)$. Then there is a unique linear map $\overline{f} : V/H \to W$ with $\overline{f}([v]) = f(v)$ for all $v \in V$.

The subspaces of V/H correspond exactly to the subspaces of V containing H.

$$f : V \to W$$

$$H = \ker(f)$$

$$\overline{f} : V/H \longrightarrow W$$

$$\overline{f}([v]) = f(v)$$

$$\overline{f}([v]) = 0$$

$$f(v) = 0$$

$$v \in H$$

$$[v] = [0]$$

$$\ker(\overline{f}) = 0$$

$f : V \longrightarrow W$

$\ker(f) = H$

$K \subseteq H$

$\bar{f} : V/K \longrightarrow W$

$\bar{f}([v]) = f(v)$

$[v] = [w] \iff v = w + k \quad k \in K$

$f(v) = f(w) + f(k) = f(w)$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\pi"] & W \\ V/K \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$K \subseteq \ker(f)$

$\pi : V \longrightarrow V/H$ onto with kernel H

$W \leq V/H$

$\pi^{-1}(W) \leq V$ contains H $\quad 0 \in W$

$\{v \in V\}$ $\pi(v) = 0$ is H.

$W \leq V \qquad \pi : V \longrightarrow V/W$

Complexes

Definition. A *complex* of vector spaces is a sequence W^i with linear maps

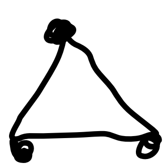
$$\cdots \to W^i \xrightarrow{d_i} W^{i+1} \xrightarrow{d_{i+1}} W^{i+2} \to \cdots$$

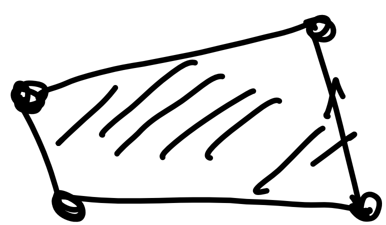
satisfying $d_{i+1}d_i = 0$ for all i. The complex is *bounded* if all but finitely many W^i are zero.

Let $Z^i = \ker(d_i)$ (the **cocycles**) and $B^i = d_{i-1}(W^{i-1})$ (the **coboundaries**). The i^{th} **cohomology** is $H^i(W^\cdot) = Z^i/B^i$.

W^0 = Functions Vertices $\to \mathbb{R}$

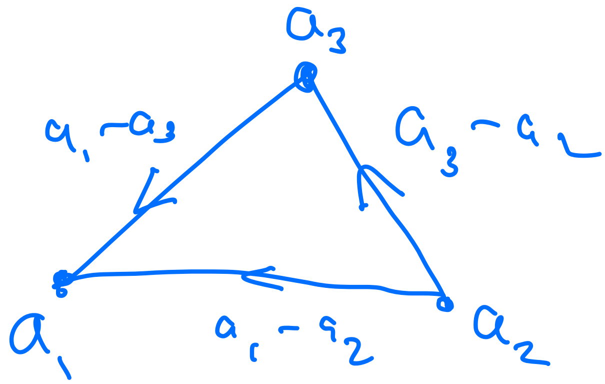
W^1 = Functions Edges $\to \mathbb{R}$





W^0 3 dim'l

$0 \to W^0 \to W \to 0$ df(a→b)

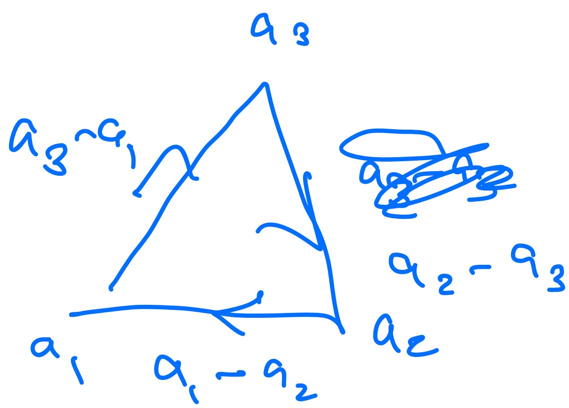


Z^0 1 dim'l

constant fns on vertices.

$Z^1 = W^1$.

Z^1



$B^0 = (0)$

Given x, y, z $\quad x + y + z = 0$

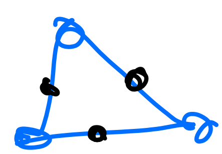
Find a_1, a_2, a_3 so that

$$\boxed{\begin{aligned} a_1 - a_2 &= x \\ a_2 - a_3 &= y \\ a_3 - a_1 &= z \end{aligned}}$$

3 vertices $\quad \dim W^0 = 3$

3 edges $\quad \dim W^1 = 3$

$$\chi = 3 - 3 = 0$$

$H^0 = 1$ dim'l in  case

$f(b) \; f(a)$

Euler Characteristic

For a bounded complex of finite dimensional spaces, the **Euler characteristic** is

$$\chi(W^\cdot) = \sum_i (-1)^i \dim(W^i).$$

Proposition.

$$\chi(W^\cdot) = \sum_i (-1)^i \dim H^i(W^\cdot).$$

$\dim H^0 - \dim H^1 = 0$

$(\dim W^0 - \dim W^1) = 0$

$\boxed{\dim H^1 = 1}$

Exact Complexes

Definition. A complex is **exact** if all its cohomology groups are zero; equivalently $\ker(d^i) = \mathrm{image}(d^{i-1})$ for all i.

A **short exact sequence** is a 5-term exact complex

$$0 \to A \to B \to C \to 0,$$

so $\underline{A \to B \text{ is injective}}$, $\underline{B \to C \text{ is surjective}}$, and $C \cong A/B$.

$A \cong \ker B \to C$

so $C = A/B$

$$\begin{tikzcd} B \arrow[r] \arrow[d, dash] & C \\ B/A \arrow[ur] & \end{tikzcd}$$

Homology

For a complex indexed the other way, $d_i : W_i \to W_{i-1}$:

- $Z_i = \ker(d_i)$ is the space of i-**cycles**.
- $B_i = \mathrm{image}(d_{i+1})$ is the space of i-**boundaries**.
- $H_i(W^\cdot) = Z_i / B_i$ is the i^{th} **homology group**.

Maps of Complexes

Definition. A map of complexes $f : \{W^\cdot\} \to \{V^\cdot\}$ is a family $f_i : W^i \to V^i$ with $f_{i+1} \circ d_i = d_i \circ f_i$.

Such an f carries cocycles to cocycles and coboundaries to coboundaries, and therefore induces the **induced map on cohomology**

$$\overline{f} : H^i(W^\cdot) \to H^i(V^\cdot).$$

$$\begin{tikzcd} V^i \arrow[r, "d_i"] \arrow[d, "f^i"'] & V^{i+1} \arrow[d, "f^{i+1}"] \\ W^i \arrow[r, "d_i"] & W^{i+1} \end{tikzcd}$$