Linear Maps, Change of Basis, Similarity

Math 5250 — Week 1

Coordinates Relative to a Basis

Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be linear, with bases $E = \{e_1, \ldots, e_n\}$ of $\mathbf{R}^n$ and $F = \{f_1, \ldots, f_m\}$ of $\mathbf{R}^m$.

- Writing $x = \sum x_i e_i$, there are unique scalars $y_1, \ldots, y_m$ with $L(x) = \sum y_i f_i$.
- In coordinates, $\boxed{x = [x]_E}$ and $L(x) = [L(x)]_F$.
- L is determined by its values $L(e_i)$ on the basis E.

$x = \sum x_i e_i$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x]_E$

The Matrix of a Linear Map

Definition. $[L]_E^F$ is the $m \times n$ matrix whose i-th column is $L(e_i)$ expressed in the basis F.

By construction,

$$[L(x)]_F = [L]_E^F \, [x]_E .$$

Ax

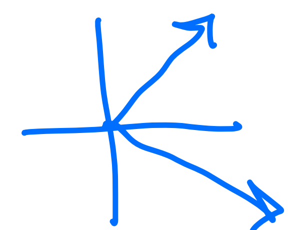
$$\begin{bmatrix} [L(e_1)]_F & \cdots & [L(e_n)]_F \end{bmatrix}$$

$\parallel$ written

Example $L = \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$

$$E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$F = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$



$x + y = a$

$x - y = b$

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$[\mathrm{id}]_E^F$$

$$\mathrm{id}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathrm{id}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[\mathrm{id}]_E^F = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\frac{5}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Change of Basis

$[id]_E^F \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$

Let E', F' be other bases of $\mathbf{R}^n$, $\mathbf{R}^m$, and write $e_j = \sum_i a_{ij} e'_i$, giving a matrix $A = (a_{ij})$.

- $A[x]_E = [x]_{E'}$, so $A = [I]_E^{E'}$, the matrix of the identity map from basis E to basis E'.
- $[I]_E^{E'} = \left([I]_{E'}^{E}\right)^{-1}$.
- The matrices of L in the two pairs of bases are related by

$$[L]_E^F = [I]_{F'}^F \, \underbrace{[L]_{E'}^{F'}}_{m \times n} \, [I]_E^{E'}.$$

$\mathbb{R}^n \qquad E, E' \qquad E = (e_1, \ldots, e_n)$

$E' = (e'_1, \ldots, e'_n)$

$e_1 = \sum a_{i1} e'_i$

$\vdots$

$e_n = \sum a_{in} e'_i \qquad Id : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & & a_{nn} \end{bmatrix} = [Id]_E^{E'} = \left[[L(e_1)]_{E'} \cdots [L(e_n)]_{E'} \right]$$

Similarity of Matrices

$A = PBP^{-1}$

For $L : \mathbf{R}^n \to \mathbf{R}^n$ and bases E, E',

$$[L]_E^E = P[L]_{E'}^{E'}P^{-1}, \qquad P = [I]_{E'}^E.$$

$P^{-1} = [I]_E^{E'}$

Definition. Square matrices A, B are **similar** if $A = PBP^{-1}$ for some invertible P.

- All matrices $[L]_E^E$ representing a single linear map L (in different bases) are similar.
- **Proposition.** Similar matrices have the same rank and the same nullity. Rank and nullity are therefore well-defined invariants of L, independent of the basis.

X

$\underbrace{PXP^{-1}}$

$\mathrm{rk}\, X = \dim \operatorname{span} \text{cols of } X$

$P[XP^{-1}]$

col rk = col rk X

$r_1, \ldots, r_k$ basis for span col space

$Pr_1, \ldots, Pr_k$

$\mathrm{rk}(L) = \dim \text{image}$

$\mathrm{null}(L) = \dim \ker = \{v \mid Lv = 0\}$

dim rk(L) + null(L) = # col of L

Injective, Surjective, and Isomorphisms

- $L : V \to W$ is **injective** if and only if its nullity is 0.
- $L : V \to W$ is **surjective** if and only if its rank equals $\dim W$.
- An **isomorphism** $L : V \to W$ is a linear map with a linear inverse; equivalently, L is bijective, which forces $\dim V = \dim W$.

L injective $\iff$ if $L(x) = L(y)$ then $x = y$

$L(x) = L(y)$

$L(x) - L(y) = 0$

$L(x-y) = 0$

if nullity $= 0$ then $x - y = 0$ so $x = y$

if not $\exists v \neq 0$ and $L(v) = 0$

$L(x+v) = L(x)$

Invertibility Criteria

Proposition. A linear map $L : \mathbf{R}^n \to \mathbf{R}^n$ is invertible if and only if its nullity is 0.

- An $n \times n$ matrix A is invertible if and only if $\mathrm{rank}(A) = n$, equivalently $\mathrm{nullity}(A) = 0$.