Rank, Nullity, and Row Reduction

Math 5250 — Week 1

Matrices

An $n \times m$ matrix over $\mathbf{R}$ is an array with n rows and m columns; the set of these is $M_{n\times m}(\mathbf{R})$ (similarly $M_{n\times m}(\mathbf{C})$ for complex arrays).

- $M_{n\times m}(\mathbf{R})$ is a vector space of dimension nm, with addition and scalar multiplication entry by entry: $(A+B)_{ij} = a_{ij} + b_{ij}$ and $(\lambda A)_{ij} = \lambda a_{ij}$.
- Elements of $\mathbf{R}^n$ may be viewed as $n \times 1$ column vectors or $1 \times n$ row vectors; by convention we use $n \times 1$ column vectors.

Matrix Multiplication

For A an $n \times m$ matrix and B an $m \times k$ matrix, the product $C = AB$ is $n \times k$ with entries

$(AB)_{ij} = i^{th}$ row of A $\cdot$ j^{th} col of B

$$\begin{array}{c} \\ n \end{array}\overset{m}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ & & & \vdots \\ & & & a_{nm} \end{pmatrix}} \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ b_{21} & & \\ \vdots & & \\ b_{m1} & & \end{pmatrix}$$

$$c_{ij} = \sum_{r=1}^{m} a_{ir} b_{rj}.$$

- Matrix multiplication is associative: $(AB)C = A(BC)$.
- It distributes over addition: $(A + B)C = AC + BC$ and $A(B + C) = AB + AC$.
- It is **not** commutative, even when both AB and BA are defined.

j^{th} col of AB is the sum of the cols of A weighted
by the els of j^{th} col of B

3 ways to think about matrix multiplication

$A = (a_{ij}) \qquad B = (b_{jk}) \qquad C = AB$

C_{ij} – dot product of row i of A with column j of B

rth column of C is the sum of the columns of A weighted by the entries in the rth col of B

$$\begin{pmatrix} | & | & | & \cdots \\ | & | & | & \\ | & | & | & \end{pmatrix}\begin{pmatrix} \cdots & b_{1r} & \\ & b_{2r} & \vdots \\ & \vdots & \\ & b_{kr} & \end{pmatrix}$$

rth row of C as the sum of the rows of B weighted by the rth row of A

$$\begin{pmatrix} \vdots & & \\ a_{r1} & a_{r2} & \cdots \end{pmatrix}\begin{pmatrix} \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} \end{pmatrix}$$

Linear Maps

Definition. $L : \mathbf{R}^m \to \mathbf{R}^n$ is **linear** if $L(x+y) = L(x) + L(y)$ and $L(ax) = aL(x)$.

$n \times m \cdot m \times 1 \rightsquigarrow n \times 1 \in \mathbb{R}^n$

- For an $n \times m$ matrix A, $L_A(x) = Ax$ defines a linear map $\mathbf{R}^m \to \mathbf{R}^n$.
- If A is $n \times m$ and B is $k \times n$, then $L_B \circ L_A = L_{BA}$.
- Everything above holds with $\mathbf{R}$ replaced by $\mathbf{C}$.

A: $n \times m$

B: $k \times n$

$L_A : \mathbb{R}^m \to \mathbb{R}^n$

$L_B : \mathbb{R}^n \to \mathbb{R}^k$

$L_B L_A : \mathbb{R}^m \to \mathbb{R}^k$

$k \times m$ matrix BA

$= L_{BA}$

Key Subspaces and Invariants of a Matrix

Definition. The **column space** of $A \in M_{n\times m}(\mathbf{R})$ is the subspace of $\mathbf{R}^n$ spanned by the columns of A; it equals the image of L_A. The **column rank** is its dimension.

$x \to Ax$ $\quad A \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ get wtd sum of cols of A by x_i

Definition. The **nullspace** (or **kernel**) of A is $\{x \in \mathbf{R}^m : Ax = 0\}$, a subspace of $\mathbf{R}^m$; the **nullity** is its dimension.

Definition. The **row space** of A is the subspace of $\mathbf{R}^m$ spanned by the rows of A; the **row rank** is its dimension.

Elementary Row Operations and Elementary Matrices

Three **elementary row operations**:

- Swap two rows.
- Multiply a row by a nonzero scalar.
- Replace $A_{j\cdot}$ by $A_{j\cdot} - \lambda A_{i\cdot}$.

$Ax = b$

given b, A
find x

$$\underbrace{\begin{pmatrix} A \end{pmatrix}}_{n \times m}^{} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = b$$

n eqns
m unknowns

Each corresponds to an **elementary matrix** E, with $A' = EA$; elementary matrices are invertible.

Proposition. If A' is obtained from A by elementary row operations, then A' and A have the same row space, the same nullspace, and the same column rank.

$$A'x = 0 \implies EAx = 0$$

$$(E^{-1})EAx = 0$$

$$Ax = 0$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

swap rows
1, 4

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & & & \\ 1 & 0 & 0 & \cdots & & \end{pmatrix}$$

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scale

$$\begin{array}{r c} & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \\ \text{row}_i \rightarrow & \end{array}$$

A multiplies row i by a
$a \neq 0$

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replace row i by row $i - \lambda$ row j

jth position

$$\begin{array}{r c} & \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix} \\ \text{row } i & \end{array}$$

Row Reduced Echelon Form

Definition. A is in **row reduced echelon form** if:

a. All-zero rows are at the bottom.
b. The first nonzero entry ("pivot") of each nonzero row is 1.
c. Pivots move down and to the right.
d. A pivot is the only nonzero entry in its column.

Proposition. Every matrix A can be transformed into row reduced echelon form by a sequence of row operations.

$$\begin{array}{cc} & \begin{matrix} & & 5 & & \end{matrix} \\ 3 & \begin{pmatrix} 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Rank and Nullity from RREF

Suppose A is $n \times m$, in row reduced echelon form, with r pivots.

Proposition.

- The row rank and column rank both equal r.
- The nullity equals $m - r$: for each non-pivot column j, a vector $x^{(j)} \in \ker A$ is obtained by putting 1 in position j, 0 in the other non-pivot positions, and $-A_{i,j}$ in each pivot position p_i. These $m - r$ vectors form a basis of $\ker A$.

row rank = col rank = # pivots

nullity = # cols − # pivots

$\boxed{\text{rank} + \text{nullity} = \#\text{ columns}}$

The Rank–Nullity Corollary

Corollary. For any matrix A with m columns:

- The row rank and column rank of A agree; this common value is the **rank** of A.
- $\mathrm{rank}(A) + \mathrm{nullity}(A) = m$.

$n \times m$ matrix
has rank at most the minimum
of n, m.

Algorithm: Testing Membership in the Range

To decide whether v is in the range of A:

- Form the augmented matrix $A' = [A, v]$.
- Row reduce A' to echelon form.
- If the last column contains a pivot, v is **not** in the range; otherwise it **is**.

$$\overset{n\times n}{A}\ \overset{n\times n}{B} = I$$

$$\begin{array}{cc} & 2n \\ n & \begin{bmatrix} A & I \end{bmatrix} \end{array}$$