Vector Spaces, Subspaces, and Dimension

Math 5250 — Week 1

Real and Complex Vectors

An n-dimensional real vector is an n-tuple $(x_1, \ldots, x_n)$ with $x_i \in \mathbf{R}$; if the x_i are complex, it is an n-dimensional complex vector.

- Notation: $\mathbf{R}^n$, $\mathbf{C}^n$.
- Visualize $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ as an arrow from the origin to the point $(x_1, \ldots, x_n)$.
- Elements of $\mathbf{R}$ or $\mathbf{C}$ are called *scalars*.

Operations

$\mathbf{R}^n$ (or $\mathbf{C}^n$) has two operations:

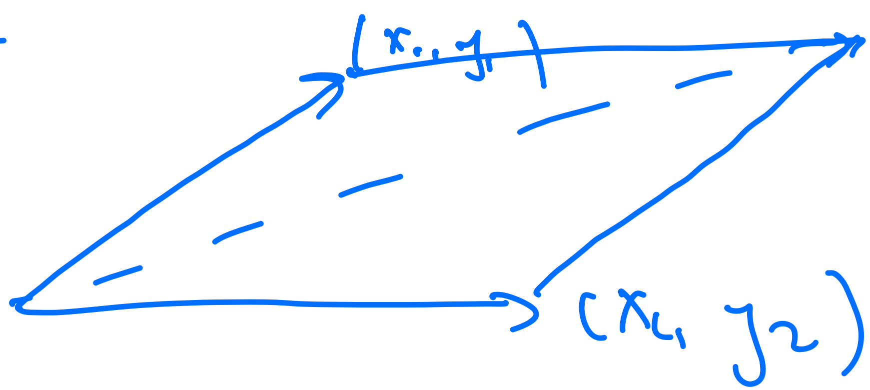
Addition. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are in $\mathbf{R}^n$ (or $\mathbf{C}^n$), then

$$x + y = (x_1 + y_1, \ldots, x_n + y_n).$$

Scalar multiplication. If $a \in \mathbf{R}$ (or $\mathbf{C}$) and $y = (y_1, \ldots, y_n)$, then

$$ay = (ay_1, \ldots, ay_n).$$

$\mathbb{R}^2$



Axioms

These operations satisfy, for $x, y \in \mathbf{R}^n$ and $a, b \in \mathbf{R}$ (likewise for $\mathbf{C}$):

1. Addition is commutative and associative; $x + e = e + x = x$ where $e = (0, \ldots, 0)$.
2. Distributivity: $(a + b)x = ax + bx$ and $a(x + y) = ax + ay$.
3. Associativity of scalars: $a(bx) = (ab)x$.
4. $1x = x$ and $0x = e$.

Vector Spaces

Definition. A *vector space* over $\mathbf{R}$ is a set V with

- **Addition:** $x, y \in V \implies x + y \in V$;
- **Scalar multiplication:** $a \in \mathbf{R},\ x \in V \implies ax \in V$,

satisfying the axioms above; in particular V has an element e with $x + e = e + x = x$ for all x.

A vector space over $\mathbf{C}$ is the same, with scalars drawn from $\mathbf{C}$. $\mathbf{R}^n$ and $\mathbf{C}^n$ are the basic examples.

Subspaces: Definition and Examples

Definition. A subspace W of V is a subset that is itself a vector space under the operations inherited from V.

Example. $W = \{(x_1, x_2, 0)\} \subset \mathbf{R}^3$ is a copy of $\mathbf{R}^2$ sitting inside $\mathbf{R}^3$, so it is a subspace.

Example. $W = \{x \in \mathbf{R}^n : x_1 + \cdots + x_n = 0\}$ is a subspace.

$$\underline{(x_1, \ldots, x_n)} + \underline{(y_1, \ldots, y_n)}$$

$$= \underline{(x_1 + y_1, \ldots, x_n + y_n)}$$

Subspaces: A Non-Example, and a Test

Example. $W = \{x \in \mathbf{R}^n : x_1 + \cdots + x_n = 1\}$ is **not** a subspace (it doesn't contain e).

Proposition (Subspace Test). $W \subset V$ is a subspace provided

- $e \in W$,
- $u, v \in W \implies u + v \in W$,
- $u \in W,\ a \in \mathbf{R} \implies au \in W$.

The same statement holds with $\mathbf{C}$ in place of $\mathbf{R}$.

Four Crucial Definitions

- **Span** of a set of vectors
- **Linear independence**
- **Basis**
- **Dimension**

Span of a Set of Vectors

Definition. For $S = \{v_1, \dots, v_k\} \subset V$, the **span** of S is either of:

1. all vectors $\sum_{i=1}^{k} a_i v_i$, as $(a_1, \dots, a_k)$ ranges over $\mathbf{R}^k$; W
2. the intersection of all subspaces of V containing S.

① $\{\sum a_i v_i\}$ is a subspace containing all v_i

$\sum a_i v_i + \sum b_i v_i = \sum (a_i + b_i) v_i$

$c(\sum a_i v) = \sum (c a_i) v_i$

Let U be a subspace containing S

then $\sum a_i v_i \in U$ for all a_i by closure

so $W \subseteq U \Rightarrow W \subseteq \bigcap U$ W is one of the "U"s.

Linear Independence

Definition. $v_1, \ldots, v_n \in V$ are **linearly independent** if the only solution of

$$\sum_{i=1}^{n} a_i v_i = 0$$

is $a_1 = \cdots = a_n = 0$.

Linear Independence: Equivalent Characterizations

P

$S = \{v_1, \ldots, v_n\}$ is linearly independent if and only if

- the span of every **proper** subset of S is a **proper** subspace of $\mathrm{Span}(S)$; equivalently, Q
- **no** v_j can be written as a combination of the others: $v_j = \sum_{i \neq j} a_i v_i$.

$\neg P \Rightarrow \neg Q$

Span(S)

suppose S NOT independent.

$\sum a_i v_i = 0$ NOT all $a_i = 0$.

Say $a_1 \neq 0$. $v_1 = -\sum_{i=2} \frac{a_i}{a_1} v_i$

$v_1 \in \mathrm{Span}(\{v_2, \ldots, v_n\})$

So $\mathrm{Span}\{v_2, \ldots, v_n\} = \mathrm{Span}(S)$

$\neg Q$: $\exists S' \subsetneq S$ with $\mathrm{Span}(S') = \mathrm{Span}(S)$

Say $S' = S \setminus \{v_1\}$ $\mathrm{Span}(S') = \mathrm{Span}(S) \Rightarrow v_1 \in \mathrm{Span}(S')$

Basis

Definition. $S \subset V$ is a **basis** of V if $\mathrm{Span}(S) = V$ and S is linearly independent.

Dimension

Theorem. Every vector space has a basis.

Theorem. Any two bases X, Y of a vector space V have the same cardinality.

Definition. The **dimension** of V is the cardinality of any basis of V.

- V is **finite-dimensional** if it has a finite basis.
- Otherwise V is **infinite-dimensional**.

Zorn's Lemma

Start with a partially ordered set X, rel'n $a \leq b$ and $b \leq c$ then $a \leq c$

$a \leq a$

Chain:

$a_1 \leq a_2 \leq a_3 \leq \cdots$

if every chain has an upper bound

given $a_1 \leq a_2 \leq \cdots \leq \cdots$

there is x with $a_i \leq x \;\forall i.$

then there is a maximal element in the set.

i.e. $\exists M$ with $a \leq M \;\forall a \in X.$

$a_1, a_2 \ldots$ all linearly independent

$a_1 \subseteq a_2 \subseteq \cdots$

$\bigcup a_i$ is still linearly independent

so there is a maximal lin. indep set. $\to$ basis

$\{ (a_1, a_2, \ldots) \mid$ all but finitely many $a_i = 0 \}$

$\{ (a_1, a_2, \ldots) \}$