9. Tensor Products

Tensor Products I: Extension of Scalars

Restriction of scalars

Suppose R and S are rings with unity (but not necessarily commutative), that we have ring homomorphism $f : R \to S$ and that M is an S module. Then M is an R module by "restricting scalars" so that $r \cdot m = f(r)m$.

- Any $\mathbb{R}$-vector space is a $\mathbb{Q}$ vector space.
- Any vector space over $\mathbb{Z}/p\mathbb{Z}$ is a module over $\mathbb{Z}$.
- There is a ring map $\mathbb{C} \to M_2(R)$ sending

$$i \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  and so any left module over $M_2(\mathbb{R})$ can be viewed as a complex vector space. There are other elements in $M_2(\mathbb{R})$ satisfying $x^2 + 1 = 0$, and so there are lots of ways to view a left module over $M_2(\mathbb{R})$ as a $\mathbb{C}$-vector space.

Extension of scalars

Suppose that $f : R \to S$ is a map of rings with unity, and M is an R-module. Is there a way to make M into an S-module? Maybe a better way to say it is: can we find a "smallest" S-module N together with a map $M \to N$?

Example: Suppose that V is a finite dimensional real vector space. Choose a $v_1, \ldots, v_n$ for V. So V is isomorphic to $\mathbb{R}^n$ using this basis. So we can think of V as inside $\mathbb{C}^n$. If we chose a different basis, we'd get a different map $V \to \mathbb{C}^n$, but the two maps would be related by a change of basis transformation so in some sense these are "isomomorphic".

Extension of scalars (More examples)

Example: Suppose that M is an abelian group (hence a $\mathbb{Z}$-module). Can we embed M in a $\mathbb{Q}$-module? Sometimes, yes: if M is $\mathbb{Z}^n$ for some n, then M embeds in $\mathbb{Q}^n$. On the other hand if M is finite, there are no maps from $M \to \mathbb{Q}^n$ for any n. If M is a mixture of free and torsion parts, we can embed the free part of M in $\mathbb{Q}^n$ but not the torsion part.

Example: If M is an abelian group, can we embed M into a vector space over $\mathbb{Z}/p\mathbb{Z}$ – here the map $R \to S$ is the map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$? Sometimes yes – if M is p-torsion, we can do it, but for general M no.

Universal approach

To study this in general we take a (left) R-module M and a ring map $f : R \to S$ and ask: how can we make an S-module out of M and the map $f : R \to S$?

An S-module structure on M means we need a map

$$S \times M \to M$$

satisfying the axioms

- $(s, (m_1 + m_2)) = (s, m_1) + (s, m_2)$
- $(s_1 + s_2, m) = (s_1, m) + (s_2, m)$
- $(sr, m) = (s, rm)$ for $r \in R$.

Extension of scalars via tensor product

Our strategy is to make an abelian group whose elements are pairs (s, m) modulo the relations above. The equivalence classes of this abelian group are written $s \otimes m$ (or, in cases where we need more context, $s \otimes_R m$ or even $s \otimes_f m$). THe group itself is called $S \otimes_R M$. So the following rules hold:

- $(s_1 + s_2) \otimes m = s_1 \otimes m + s_2 \otimes m$.
- $s \otimes (m_1 + m_2) = s \otimes m_1 + s \otimes m_2$
- $sr \otimes m = s \otimes rm$.

A typical element of $S \otimes M$ is a *sum* of the form $\sum_{i=1}^{n} s_i \otimes m_i$. It is an S-module via multiplication by S on the first factor.

We have a map $M \to S \otimes_R M$ given by $m \mapsto 1 \otimes m$.

Important: The elements $s \otimes m$ belong to a quotient group and so the representation of an element as a sum of "simple tensors" $s \otimes m$ need not be unique! In fact it's quite possible for $s \otimes m$ to be zero.

Some examples

Suppose that $M = \mathbb{Z}^n$ and $R \to S$ is $\mathbb{Z} \to \mathbb{Q}$. Then $\mathbb{Q} \otimes M$ consists of sums of elements $a \otimes m$. But $a = \frac{x}{y}$ where $x \in \mathbb{Z}$, so we can write $a \otimes m = \frac{1}{y} \otimes xm$. $\mathbb{Q} \otimes M$ is isomorphic to $\mathbb{Q}^n$.

Suppose that M is a finite group of order n. Then any element of $\mathbb{Q} \otimes M$ can be written $a \otimes m$ where $a \in \mathbb{Q}$. But $a = n(a/n)$. Therefore

$$a \otimes m = (a/n)n \otimes m = (a/n) \otimes nm = 0$$

so $\mathbb{Q} \otimes M$ is the zero module.

The universal property

The idea is that $S \otimes M$ is the *smallest* S module containing M, where the action of R comes via the map $R \to S$. In other words, if L is any other S module and there is an R-module map $f : M \to L$, then there is a unique S-module map $\phi : S \otimes M \to L$ so that this triangle commutes:

$$\begin{tikzcd} M \arrow[r, "1 \mapsto 1 \otimes m"] \arrow[dr, "f"'] & S \otimes M \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

More on the universal property

If $\iota : M \to S \otimes M$ is the map $m \mapsto 1 \otimes m$, then $M/\ker(\iota)$ maps injectively in $S \otimes M$. This is the "largest" quotient of M which embeds into an S-module.

Example: Let G be a finitely generated abelian group. Then G is isomoprhic to $\mathbb{Z}^n \oplus T$ where T is a finite group. If we want to map G to a $\mathbb{Q}$-vector space, the kernel has to include T. And in fact the kernel of ι is T. Further, the $\mathbb{Q}$-dimension of the vector space $\mathbb{Q} \otimes G$ is the rank of the free part of G.

More examples

Example: Let M be an R module and let $f : R \to R/I$ be the quotient map. Then $R/I \otimes M$ is isomorphic to M/IM. First notice that if $x \in IM$, then $1 \otimes x = 1 \otimes im = i \otimes m = 0$, so IM is in the kernel of ι. Therefore we have a map $M/IM \to R/I \otimes M$. We have a map in the opposite direction $R/I \otimes M \to M/IM$ given by $(r + I) \otimes m \mapsto rm + IM$. So if G is a finite abelian group, then $\mathbb{Z}/p\mathbb{Z} \otimes G$ is G/pG which is zero if G has no p-torsion.

Example: If V is a vector space over F of dimension n, and $F \to E$ is a field extension, then $E \otimes V$ is an n-dimensional vector space over E.

Tensor products of modules

The commutative case

Assume for the moment that R is a *commutative* ring with unity. Suppose that M and N are R-modules. If L is yet another R-module, a bilinear map

$$f : M \times N \to L$$

is a map that is linear in each variable separately and also satisfies $f(rm, n) = f(m, rn)$ for $r \in R$. The tensor product $M \otimes_R N$ of M and N is the free abelian group on pairs (m, n) modulo the relations:

- $(m_1 + m_2, n) = (m_1, n) + (m_2, n)$
- $(m, n_1 + n_2) = (m, n_1) + (m, n_2)$
- $(rm, n) = (m, rn)$

The equivalence class of the pair (m, n) is written $m \otimes n$. $M \otimes N$ is an R module via the action $r(m \otimes n) = (rm \otimes n) = (m \otimes rn)$ and

$$r(\sum m_i \otimes n_i) = \sum r(m_i \otimes n_i).$$

Universal property

If $f : M \times N \to L$ is bilinear, then we can defined $\overline{f} : M \otimes N \to L$ by $\overline{f}(m \otimes n) = f(m, n)$. This is well defined and it converts a bilinear map into a module homomorphism. To go in the other direction, there is a map $B : M \times N \to M \otimes N$ which is bilinear, sending $(m, n) \to m \otimes n$. This is the "universal" bilinear map. The universal property says that, if $f : M \times N \to L$ is any bilinear map, there is a unique module homomorphism $\overline{f} : M \otimes N \to L$ such that $\overline{f}B = f$.

$$\begin{tikzcd}
M \times N \arrow[r, "B"] \arrow[dr, "f"'] & M \otimes N \arrow[d, "\overline{f}"] \\
& L
\end{tikzcd}$$

Tensor product of vector spaces

Suppose that R is a field and V, W, are vector spaces over R of dimensions n and m respectively. Let $v_1, \ldots, v_n$ be a basis for V and $w_1, \ldots, w_m$ a basis for W.

If L is another F-vector space, then a bilinear map $f : V \times W \to L$ is determined by its values on all pairs (v_i, w_j).

The tensor product $V \otimes W$ is an F-vector space and is spanned by the tensors $v_i \otimes w_j$.

Now construct a bilinear map $f_{ij} : V \times W \to F$ by setting

$$f_{ij}(\sum a_s v_s, \sum b_s w_s) = a_i b_j.$$

By the universal property we have $f_{ij}(v_r \otimes w_s) = 0$ unless $r = i$ and $s = j$ in which case it is one.

Tensor product of vector spaces continued

Suppose that

$$x = \sum c_{rs} v_r \otimes w_s = 0.$$

Then $f_{ij}(x) = c_{ij} = 0$ for all pairs i, j and therefore all $c_{rs} = 0$; in other words, the $v_r \otimes w_s$ are linearly independent. Thus $V \otimes W$ is an nm dimensional F-vector space.

Endomorphisms

Let V be a vector space and let V^* be its dual. Then there is an isomorphism

$$V \otimes V^* \to \mathrm{End}(V)$$

where $(v \otimes f)(w) = f(w)v$.

The noncommutative case

Now suppose that R is a noncommutative ring. If M and N are left modules, then we have a problem defining a bilinear map $M \times N \to L$ where L is also a left module. Namely, on the one hand, we would need:

$$f(rsm, n) = rf(sm, n) = f(sm, rn) = sf(m, rn) = f(m, srn)$$

but on the other hand

$$f((rs)m, n) = (rs)f(m, n) = f(m, (rs)n)$$

and since sr and rs are different we can't define this consistently.

The noncommutative case continued

In the non-commutative case (with unity) we have to make some compromises:

- First, we assume M is a *right* module and N is a *left* module.
- Next, we are only going to construct an *abelian group*, not a module, from M and N.
- Finally, we are going to consider maps $f : M \times N \to L$, where L is an abelian group, that are *balanced*, meaning that $f(mr, n) = f(m, rn)$ for $r \in R$.

We create an abelian group spanned by $m \otimes n$ subject to the relations $mr \otimes n = m \otimes rn$ together with the additivity $(m + m') \otimes n = m \otimes n + m' \otimes n$ and similarly for N.

This is the tensor product of the modules M and N – remember it is an abelian group, NOT an R-module in general.

Universal property

$M \otimes N$ still satisfies a universal property. Call a map $f : M \times N \to L$, where M is a right R-module, N is a left R-module, and L is an abelian group, *balanced* if f is additive in M and N separately and satisfies $f(mr, n) = f(m, rn)$ for all $r \in R$.

Then given such a balanced map from $M \times N$ to L, there is a unique map of abelian groups $\phi : M \otimes N \to L$ such that the following diagram commutes.

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "f"'] & M \otimes N \arrow[d, "\phi"] \\ & L \end{tikzcd}$$

Here the map $M \times N \to M \otimes N$ is the expected one: $(m, n) \mapsto m \otimes n$.

As is always the case, the universal property characterizes the tensor product up to isomorphism.

Bimodules

Now suppose that S and R are rings with unity and that M is simultaneously a left S-module and a right R module, so that $(sm)r = s(mr)$. Such an object M is called an (S, R)-bimodule.

For example, suppose $R = M_2(F)$, $S = F$, and M is the space F^2 viewed as row vectors with R acting on the right as matrix multiplication and S on the left as scalar multiplication.

Tensor product of bimodules

If N is a left R module, we can form the tensor product $M \otimes_R N$ which is an abelian group; but we can furthermore let S act by $s(m \otimes n) = (sm \otimes n)$.

This makes $M \otimes_R N$ into a left S-module. (If N is an (R, S)-bimodule so that R acts on the left and S on the right, then $M \otimes_R N$ is a right S-module.)

If R is commutative, and M is a left R module, it is also a right R-module via $(mr) = rm$. So it is automatically an (R, R)-bimodule. This is how $M \otimes_R N$ is automatically an R-module if R is commutative.

General Properties

Tensor product of maps

If $f : M \to M'$ and $g : N \to N'$ are maps of right/left R-modules, then $f \otimes g : M \otimes N \to M' \otimes N'$ defined by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ is a well defined group homomorphism.

If M and M' are (S, R) bimodules and f and g are S-module homomorphismsm then $f \otimes g$ is an S-module homomorphism. (If R is commutative all this is automatic).

Further, provided everything makes sense, $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$.

The Kronecker Product

If L and M are matrices giving linear maps from $\mathbb{R}^n$ to $\mathbb{R}^{n'}$ and $\mathbb{R}^m$ to $\mathbb{R}^{m'}$.

Then the tensor product of these maps is a linear map from $\mathbb{R}^n \otimes \mathbb{R}^m$ to $\mathbb{R}^{n'} \otimes \mathbb{R}^{m'}$.

The standard bases of $\mathbb{R}^k$ give us bases $e_i \otimes e_j$ of $\mathbb{R}^n \otimes R^m$ and $\mathbb{R}^{n'} \otimes \mathbb{R}^{m'}$. Thus the tensor product is represented as an $nm \times n'm'$ matrix. This is called the Kronecker Product of the matrices L and M.

Associativity

The tensor product is associative in the sense that $(M \otimes_R N) \otimes_T L$ is isomorphic to $M \otimes_R (N \otimes_T L)$ provided that M is a right R module, N is an (R, T) bimodule, and L is a left T-module. If R and T are commutative this is automatic.

First check that both versions of the tensor product make sense.

Notice that $N \otimes_T L$ is a left R-module and $M \otimes_R N$ is a right T-module.

For fixed $l \in L$, the map $(m, n) \mapsto m \otimes (n \otimes l)$ is R-balanced, so there is a well-defined map

$$M \otimes_R N \to M \otimes_R (N \otimes_T L)$$

This gives a well-defined map

$$M \otimes_R N \times L \to M \otimes_R (N \otimes_T L)$$

Commutativity

If R is commutative, the tensor product is commutative in the sense that $M \otimes N$ is isomorphic to $N \otimes M$.

Distributive law

$(M \oplus M') \otimes N$ is isomorphic to $(M \otimes N) \oplus (M' \otimes N)$ and similarly if N is a direct sum. By induction this extends to finite direct sums. With care it holds for infinite direct sums.

The proof of this uses the fact that there is a well-defined balanced map

$$F : (M \oplus M') \times N \to (M \otimes N) \oplus (M' \otimes N)$$

defined by $F((m, m'), n) = (m \otimes n, m' \otimes n)$

and so we have a map

$$(M \oplus M') \otimes N \to (M \otimes N) \oplus (M' \otimes N).$$

On the other hand we have balanced maps $M \times N \to (M \oplus M') \otimes N$ sending $m \otimes n$ to $(m, 0) \otimes n$ and similarly for $M' \times N$. These give a map from $M \otimes N \oplus M' \otimes N \to (M \oplus M') \otimes N$ which is inverse to the map above.

Tensor product of algebras

If A and B are R algebras where R is commutative, then $A \otimes_R B$ is an R algebra with multiplication $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$. (remember that an R algebra is a ring in which R is mapped into the center of the ring).