7. Galois Extensions

Galois Extensions

Automorphisms

Let K/F be an extension fields and let $\mathrm{Aut}(K/F)$ be the group of field homomorphisms $\sigma : K \to K$ which are the identity when restricted to F.

Some basics:

- If $\alpha \in K$ is algebraic over F with minimal polynomial $p(x)$ over F, then $\sigma(\alpha)$ is also a root of $p(x)$.
- If $H \subset \mathrm{Aut}(K/F)$ is a subgroup, then the set $K^H = \{x \in K : \sigma(x) = x \forall \sigma \in H\}$ is a subfield of K.
- If $H_1 \subset H_2$ are subgroups of $\mathrm{Aut}(K/F)$, then $K^{H_2} \subset K^{H_1}$.

Galois Extensions

Proposition: Let E/F be the splitting field over F of some polynomial $f(x) \in F[x]$. Then

$$|\mathrm{Aut}(E/F)| \leq [E:F].$$

If $f(x)$ is separable, then this is an equality.

Definition: E/F is called a Galois extension if $|\mathrm{Aut}(E/F)| = [E:F]$. In this case the automorphism group is called the *Galois group* of the extension.

Proposition 5 says that separable splitting fields are Galois extensions.

The Galois correspondence

Let E/F be a Galois extension with Galois group G. Then there is a bijective (inclusion reversing) correspondence between:

- subfields L of E containing F
- subgroups of G

The correspondence is given by $H \to E^H$ for $H \subset G$ in one direction, and $L \to \mathrm{Aut}(E/L) \subset G$ in the other direction.

Further:

- If $L = E^H$ then E/L is Galois with group H.
- If $L = E^H$ then $[E : L] = |H|$ so E is Galois over L.
- If L is a subfield of E containing F, then $|\mathrm{Aut}(E/L)| = [E : L]$ so E is Galois over L.
- The fixed field $L = E^H$ is Galois over F if and only if H is a normal subgroup of G, and in that case $\mathrm{Aut}(L/F) = G/H$.

Some examples

- Quadratic extensions
- $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
- The splitting field of $x^3 - 2$ over $\mathbb{Q}$.
- The splitting field of $x^4 - 2$ over $\mathbb{Q}$.
- The field $\mathbb{Q}(\zeta_p)$ where ζ_p is a p^{th} root of unity.
- The fields of eigth and ninth roots of unity.
- Finite fields.

Overview of the proof

There are two "directions" we need to consider.

1. Suppose E/F is a separable splitting field extension. Then $|\mathrm{Aut}(E/F)| = [E : F]$.
2. Suppose E is a field and G is a finite group of automorphisms of E. Then the fixed field E^G satisfies $[E : E^G] = |G|$ and E/E^G is a separable splitting field.

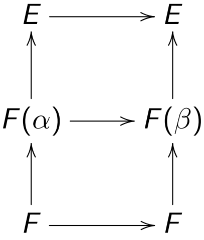
These mean together that:

- if we start with a separable extension E/F, compute its automorphism group $\mathrm{Aut}(E/F)$, and then take the fixed field in E of that group, we get a subfield of E that contains F and has $[E : E^{\mathrm{Aut}(E/F)}] = [E : F]$, so $E^{\mathrm{Aut}(E/F)} = F$.
- if we start with a field E and a group of automorphisms G, then E/E^G is a separable splitting field extension so $\mathrm{Aut}(E/E^G)$ has order $[E : E^G] = |G|$. Since G is contained in $\mathrm{Aut}(E/E^G)$, this means $\mathrm{Aut}(E/E^G) = G$.

This is the prototype of the Galois correspondence.

More on the proof - Step 1

The first assertion to consider is that, if E/F is a separable splitting field, then $|\mathrm{Aut}(E/F)| = [E : F]$. This is a consequence of the theorem on extension of automorphisms. The proof is by induction. Clearly if $E = F$ then $\mathrm{Aut}(E/F)$ is trivial and $[E : F] = 1$. Now suppose we know the result for all separable splitting fields of degree less than n and suppose E/F has degree n. Choose an element $\alpha \in E$ of degree greater than one over F and let $f(x)$ be its minimal polynomial. Let β be any other root of $f(x)$. Since E/F is a splitting field, $\beta \in E$. Consider the diagram:



More on the proof - Step 2

Now we want to show that, if G is a group of automorphisms of a field E, then E/E^G is a separable splitting field of degree $|G|$. The key tool here is a result known as *linear independence of characters*.

Lemma: Let G be a group, let L be a field, and let $\sigma_1, \ldots, \sigma_n$ be distinct homomorphisms $G \to L^\times$. Then the σ_i are linearly independent over L, meaning that if $f = \sum_{i=1}^n a_i \sigma_i$ is the zero map for some collection of $a_i \in L$, then all a_i are zero.

Proof: Suppose that the σ_i are dependent. Choose a linear relation of minimal length where all the coefficients are nonzero:

$$f = \sum a_i \sigma_i = 0$$

Let $h \in G$ such that $\sigma_1(h)$ and $\sigma_n(h)$ are different. Now $f(g) = 0$ for all $g \in G$, and also $f(hg) = 0$ for all $g \in G$ since it's the same set of elements. Therefore

More on the proof - Step 3

Now we want to prove that $[E : E^G] = |G|$. Let's use $F = E^G$ to simplify the notation. Choose a basis $\alpha_1, \ldots, \alpha_n$ for E/F and let $\sigma_1, \ldots, \sigma_m$ be the elements of G. Form $m \times n$ the matrix

$$S = \begin{pmatrix} \sigma_1(\alpha_1) & \sigma_1(\alpha_2) & \cdots & \sigma_1(\alpha_n) \\ \sigma_2(\alpha_1) & \sigma_2(\alpha_2) & \cdots & \sigma_2(\alpha_n) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_m(\alpha_1) & \sigma_m(\alpha_2) & \cdots & \sigma_m(\alpha_n) \end{pmatrix}$$

Let's first look at the row rank of this matrix. Suppose that

$$[\beta_1 \cdots \beta_m]S = 0.$$

It follows that $\sum_{i=1}^m \beta_i \sigma_i(\alpha_j) = 0$ for all α_j, and, since the α_j span E/F, we condlue $\sum_{i=1}^m \beta_i \sigma_i(x) = 0$ for all $x \in E$. By linear independence this means that all $\beta_i = 0$ and so the row rank of S is m.

More on the proof - Step 4

We finally need to verify that E/E^G is a separable splitting field. First, let $\alpha \in E$ be any element of E with minimal polynomial $q(x)$. Consider the orbit $\{\alpha_1, \ldots, \alpha_k\}$ of α under the action of G. The polynomial

$$p(x) = \prod_{i=1}^{k} (x - \alpha_i)$$

is fixed by G so has coefficients in E^G; it also has α as a root so $q(x)$ divides $p(x)$. Therefore all roots of $q(x)$ belong to E. Since every polynomial with coefficients in E^G that has a root in E splits in E, E is a splitting field over E^G.

To show separability, let $\beta_1, \ldots, \beta_n$ be a basis for E/E^G. Let $p_i(x)$ be the minimal polynomial of β_i over E^G. We've shown already that each $p_i(x)$ splits completely in E. Consider the product $f(x)$ of all the p_i and let $f_1(x)$ be its square free part (that is, the product of its irreducible factors, all to the first power). Then $f_1(x)$ is

The full proof of the correspondence

See the proof in Dummit and Foote, which basically applies our numerical result that $[E : E^G] = |G|$, the fact that E/E^G is a separable splitting field, and our existence theorem that, if E/F is a separable splitting field, then $|\mathrm{Aut}(E/F)| = [E : F]$j to obtain the correspondence.