# Sylow III

**Theorem:** Let $G$ be a finite group and $p$ a prime dividing the order of $G$. Let $n_p$ be the number of Sylow $p$-subgroups in $G$. Then $n_p$ is a divisor of $|G|$ and $n_p \equiv 1 \pmod{p}$.

$S_4$ $n_2 = 3$
$3 \mid 24$, $3 \equiv 1 \bmod 2$

$S_4$ $n_3$
divisors of 24
1, 2, 3, 4, 6, 8, 12, 24
8 3 cycles
4 Sylow 3-subgps in $S_4$
each has 2 of 8 3 cycles.

**Proof:**

① $n_p$ is a divisor of $|G|$.
By Sylow II, all Sylow $p$-subgps are conjugate.
Pick one: $P$.

$X = \underbrace{\{gPg^{-1} : g \in G\}} = \{$all Sylow $p$-subgps$\}$.

\# elts in $\{gPg^{-1} : g \in G\}$
$= [G : N(P)] \mid |G|$.

② $X = \{P_1, P_2, \ldots P_{n_p}\}$
Let $P = P_1$ act on $X$ by conjugation
$P_1$ fixes itself under conjugation
$P_1$ fixed $P_i \Rightarrow P_1 \subseteq N(P_i) \Rightarrow P_1 = P_i$
$P_i \neq P_1 \Rightarrow$ $P_i$ is not fixed by $P_1$

$X = \{P_1\} \cup X_1 \cup \cdots X_r$

$X_i$ = orbit of some $P_i$.
$|X_i| = [P : P \cap N(P_i)]$
= a power of $p$

$n_p = |X| = 1 + \sum$ multiples of $p$
$n_p \equiv 1 \bmod p$