The class equation

Definition: Let G be a ~~finite~~ group acting on a ~~finite~~ set X. We say x is a fixed point of G if $G_x = G$ – in other words if $gx = x$ for all $g \in G$.

x fixed pt $\iff$ $gx = x$ for all $g \in G$.

$\Updownarrow$

$G_x = G$.

$g \in GL_2(\mathbb{R})$

$g\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ for all g.

$\begin{bmatrix}0\\0\end{bmatrix}$ is a fixed pt for $GL_2(\mathbb{R})$ acting on $\mathbb{R}^2$.

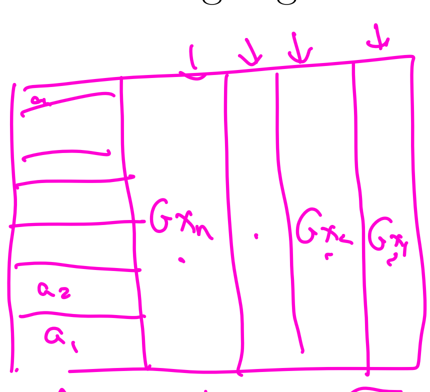
Proposition: If X is a finite set with an action by a finite group G, we have

$$|X| = |X_G| + \sum_{i=1}^{n} |Gx_i|$$

where X_G is the set of fixed points of G and the x_i are representatives for the distinct orbits of G of length greater than one.

Proof:

$|X| = |X_G| + \sum_{\text{orbits}} |\text{that orbit}|$



fixed pts

have more than one element.

$Gx = \{gx \mid g \in G\}$

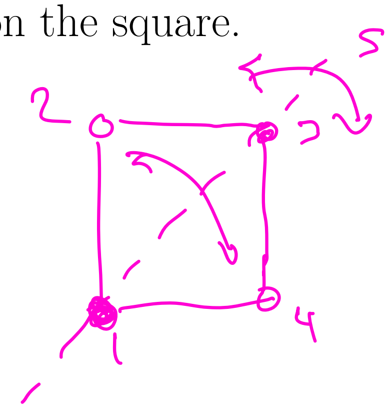
$Gx = \{y \sim x \mid \sim = g\text{-equivalence}\}$

$|Gx| = 1 \iff x$ is a fixed pt

Example: Consider the subgroup of D_4 consisting of a diagonal reflection acting on the square.

$H \le D_4$

$H = \{e, s\}$



$X = \{1, 2, 3, 4\}$

$X_H = \{1, 3\}$

$H \cdot 2 = \{2, 4\}$

$$4 = |X| = \underset{\underset{2}{\parallel}}{|X_H|} + \underset{\underset{2}{\parallel}}{|H \cdot 2|}$$

Corollary:

$$|G| = |Z(G)| + \sum_{i=1}^{n} [G : C(x_i)]$$

where $Z(G)$ is the center of G and $x_1, \ldots, x_n$ are representatives for the conjugacy classes having more than one element in G.

Proof:

$X = G$ $\qquad$ $g \cdot x = gxg^{-1}$.

Fixed pts:

$x \in X_G \Leftrightarrow g \cdot x = x \;\; \forall g \in G \Leftrightarrow gxg^{-1} = x$ for all $g \in G$

$\Leftrightarrow gx = xg$ " " "

Fixed pts for conjugation are exactly $Z(G)$.

If x is not fixed then we know that

$|Gx| = [G : G_x]$

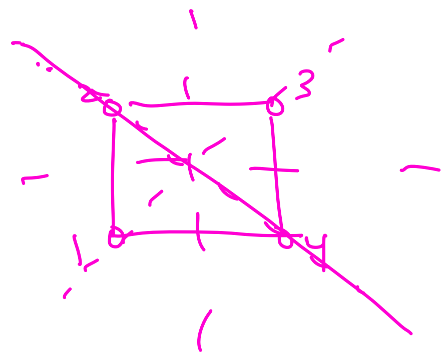
$gxg^{-1} = x \Leftrightarrow g \in \text{Centralizer}(x)$.

$\qquad\qquad\qquad\qquad \text{"} \{g \mid gx = xg\}$.

$|Gx_i| = [G : \text{Cent}(x_i)]$

↑ orbit is called a conjugacy class.

Example: Let $G = D_4$.



e

$R = (1234)$

$R^2 = \underline{(13)(24)}$

$R^3 = (1432)$

$\begin{cases} S_1 = (24) \\ S_2 = (13) \end{cases}$ $\quad S_3 = \underline{(12)(34)}$, $\quad S_4 = (14)(23)$

$Rs = sR^{-1}$

$\underline{R^2 S = SR^{-2}}$ $\qquad R^4 = e$, $\quad R^2 = R^{-2}$

$Z(D_4) = \{e, R^2\}$

$S_3 R^2 S_3^{-1} = (24)(13) = R^2$

(24)

$R^{-1}(24)R = R^{-1}R^{-1}(24) = R^{-2}(24) = R^2(24)$

$\underbrace{(13)(24)}_{R^2}(24) = (13)$

$R(12)(34)R^{-1} = R(12)R(34) = RR^{-1}(12)(34)$

$(24)(12)(34)(24) = (14)(32) = S_4$ $\qquad Z(G) = e, R^2.$

$\{S_1, S_2\}$ $\qquad \{S_3, S_4\}$, $\qquad \{R, R^3\}$

$SRS^{-1} = R^{-1} = R^3$

$\underset{8}{D_4} = \underset{2}{\{e, R^2\}} \cup \underset{2}{\{S_1, S_2\}} \cup \underset{2}{\{S_3, S_4\}} \cup \underset{2}{\{R, R^3\}}$

Corollary: The size of a conjugacy class is a divisor of the order of the group.

$|\text{conjugacy class}| \,\Big|\, |G|.$

$\#$ size of conjugacy class $= [G : G_x] \,\Big|\, |G|.$

S_4

e	1	all divide 24.
(12)	6	
$(12)(34)$	3	
(123)	8	
(1234)	6	

Corollary: A group whose order is a power of a prime has non-trivial center.

$$p^k = |G| = |Z(G)| + \sum_{\cancel{\text{orbits}}\ \text{conj class}} |\#\text{ of elts in conjugacy class}|$$

all ~~not~~ powers of p

$|Z(G)| \geq 1$

If $|Z(G)| = 1$ then

$$\underbrace{p^k}_{\text{divisible by } p} = 1 + \underbrace{\sum \text{powers of } p}_{\text{divisible by } p}$$

but that's impossible

so $|Z(G)| \geq p$.

Q $\pm 1, \pm i, \pm j, \pm k$

$\{\pm 1\} = Z(Q)$

Corollary: A group of order p^2 is abelian.

Proof: G has p^2 elts

$|Z(G)| = p$ or p^2

if $|Z(G)| = p^2$ then G is abelian.

So suppose $|Z(G)|$ has p elements

$|G/Z(G)|$ has p elts

$Z(G)$ is always normal

$gZ(G)g^{-1} = Z(G)$

$Z(G) \simeq \mathbb{Z}_p$

$G/Z(G) \simeq \mathbb{Z}_p$.

Take $aZ(G) \in G/Z(G)$.

$x \in G$ $\quad xZ(G) = a^i Z(G)$.

$x = a^i z_1$

$y \in G$ $\quad yZ(G) = a^j Z(G)$

$y = a^j z_2$

$\Rightarrow G$ is abelian

$xy = a^i z_1 a^j z_2 = a^{i+j} z_1 z_2$

$\parallel$

$yx = a^j z_2 a^i z_1 = a^{i+j} z_2 z_1 = a^{i+j} z_1 z_2$

$|G| = p^2$ then $G \simeq \mathbb{Z}_{p^2}$

or $G \simeq \mathbb{Z}_p \times \mathbb{Z}_p$