Fundamental Theorem of Finite Abelian Groups

Theorem: Let G be a finite ab~~l~~elian group. Then G is isomorphic to a product of cyclic groups of prime power order.

Examples:

- G has order 100

G finite abelian with 100 elts.

$G = G_1 \times G_2 \times \dots \times G_n$ where G_i is cyclic of prime power order

$|G_i| = p_i^{k_i}$ $\quad i = 1, \dots, n$

p_i prime

$100 = |G| = \prod_{i=1}^{n} p_i^{k_i}$

$$\begin{aligned} 100 &= 2 \cdot 2 \cdot 5 \cdot 5 \\ &= 2^2 \cdot 5 \cdot 5 \\ &= 2 \cdot 2 \cdot 5^2 \\ &= 2^2 \cdot 5^2 \end{aligned}$$

$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$

$\mathbb{Z}_4 \times \mathbb{Z}_{25}$

- G has order 27

$27 = 3^3$

$3 \cdot 3 \cdot 3$ $\quad \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

$3 \cdot 3^2$ $\quad \mathbb{Z}_3 \times \mathbb{Z}_9$

3^3 $\quad \mathbb{Z}_{27}$

Theorem: The isomorphism classes of finite abelian groups of order n correspond to sequences $d_1|d_2|\cdots d_k$ with $d_1 d_2 \cdots d_k = n$. Each such sequence corresponds to the product of cyclic groups of order d_i.

$d_i > 1$

$G = G_{d_1} \times G_{d_2} \times \cdots G_{d_k}$

- G has order 108.

$108 = 3^3 \cdot 2^2$

$d_1 = 108$	$\mathbb{Z}_{108}$
$2 \mid 54$	$\mathbb{Z}_2 \times \mathbb{Z}_{54}$
$3 \mid 36$	$\mathbb{Z}_3 \times \mathbb{Z}_{36}$
$6 \mid 18$	$\mathbb{Z}_6 \times \mathbb{Z}_{18}$
$3 \mid 3 \mid 12$	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{12}$
$3 \mid 6 \mid 6$	$\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6$

$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6$

$\underbrace{\mathbb{Z}_9 \times \mathbb{Z}_2}_{\mathbb{Z}_{18}} \times \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_3}_{\mathbb{Z}_6}$

- G has order 81.

$81 = 3^4$

$3 \mid 3 \mid 3 \mid 3$	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$
$3 \mid 3 \mid 9$	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$
$9 \mid 9$	$\mathbb{Z}_9 \times \mathbb{Z}_9$
$3 \mid 27$	$\mathbb{Z}_3 \times \mathbb{Z}_{27}$
81	$\mathbb{Z}_{81}$

Outline of proof - Main Lemmas

1. An abelian group G of order n has an element of order p, where p is prime, if and only if $p|n$.

   $p \,|\, |G| \Rightarrow \exists\, g \in G$ with order p

2. If every element of G has order a power of a fixed prime p, then the number of elements in G is a power of p. Such a group is called a finite abelian p-group.

3. If G is a finite abelian p group, then either G is cyclic or a product of a cyclic p-group and another abelian p-group.

   $G = C_{p^i} \times H$

4. If G is a finite abelian group of order nm where $\gcd(n, m) = 1$, then G is the product of the subgroups G_n and G_m consisting of elements of order dividing n and m respectively.

   $G \simeq G_n \times G_m.$

Assembly

- Start with an abelian group of order n. Use (4) to split it up into subgroups, each consisting of elements of order a power of a different prime p.

$$n = p_1^{n_1} \cdots p_k^{n_k}$$

$$G = G[p_1^{n_1}] \times G[p_2^{n_2}] \cdots$$

$$\vdots \quad G[p_i^{n_i}] = \{ g \in G \mid \text{order}(g) \text{ is a power of } p_i \}$$

- Each such factor is a finite abelian p group so it is either cyclic or splits as a cyclic factor times a smaller abelian p-group by (3)

$$G[p_i^{n_i}] \simeq C_{p_i^{s_1}} \times C_{p_i^{s_2}} \cdots \times C_{p_i^{s_k}}$$

- By repeating the previous step you reduce to the case that all factors are cyclic p-groups.