Finitely Generated Groups

Definition: Let G be a group and let $T = \{g_i : i \in I\}$ be a collection of elements of G. The smallest subgroup of G containing T is called *the subgroup generated by T* and T is said to *generate* that subgroup.

given G $\qquad$ $g_1, \dots \dots \dots$ $\qquad$ $i \in I$

H contains $g_1, g_2, \dots g_i$ $\qquad$ $i \in I$

contains $g_i \Rightarrow g_i^{-1}$

must contain any product $\underbrace{g_{i_1}^{\pm 1} g_{i_2}^{\pm 1} \cdots g_{i_k}^{\pm 1} \cdots g_{i_n}^{\pm 1}}_{\text{finite product}}$

$e \in H$.

G is called *finitely generated* if there is a finite set T that generates G.

- finite groups

  G is finite

  $T = \{g_i \mid g_i \in G\}$

  $H \leq G$ subgroup

  $H \supseteq T \Rightarrow H = G.$

- $\mathbb{Z}^k$

  $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

  $e_1 = (1,0,0)$

  $e_2 = (0,1,0)$

  $e_3 = (0,0,1)$

  $H \leq \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

  H contains e_1, e_2, e_3

  $\underbrace{ne_1, me_2, qe_3}$ $\quad n, m, q \in \mathbb{Z}.$

  $ne_1 + me_2 + qe_3 = (\underbrace{n, m, q}_{n,m,q \text{ arbitrary}})$

- wallpaper groups

  $W \leq E(2)$

  $1 \to T \to W \to F \to 1$ $\qquad W/T \simeq F.$ finite.

  $T \cong \mathbb{Z}\times\mathbb{Z}$, $t_1^a t_2^b$

  Choose $w_1, \ldots, w_n$ so that $\{w_i T\}$ are all cosets of T in $W.$

  $w = w_i t = w_i t_1^a t_2^b$

  W generated by $\{\{w_i\}, t_1, t_2\}$

Non-example: $\mathbb{Q}$ is not finitely generated.

Proof: Assume $q_1, \ldots, q_n$ generate $\mathbb{Q}$.

$q_i = \frac{a_i}{b_i}$ $\quad b_i \neq 0$ $\quad a_i, b_i \in \mathbb{Z}$.

$b_i > 0$

$\sum n_i q_i \in \mathbb{Q}$ $\qquad n_i \in \mathbb{Z}$

a rat'l number whose denominator is at most $b_1 \cdots b_n$

$H = \left\{ q \in \mathbb{Q} \mid q = \frac{a}{b_1 \cdots b_n} \quad a \in \mathbb{Z} \right\} \subsetneq \mathbb{Q}$.

H is a subgroup $\supseteq \{ q_1, \ldots, q_n \}$.

So $\{q_1, \ldots, q_n\}$ does not generate $\mathbb{Q}$.

$\mathbb{Q}$ not f.g.

$\mathbb{Z}^n$

It is true that every finitely generated subgroup of a finitely generated *abelian* group is finitely generated. But this is false in general.

Let G be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ generated by the matrices

$$a^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \qquad b^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

Let H be the subgroup of G consisting of all matrices that have ones on the diagonal. Then H is not finitely generated.

Proof: The group consists of matrices whose upper right entry is of the form $a/2^k$ for an integer a and $k \geq 0$.

$H \subseteq G$ everything in G with 1's on diagonal.

$$H = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G \right\}.$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in H \qquad \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in H \qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}.$$

$$b^{-1}ab = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \underset{H}{\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}}$$

$$b^{-n}ab^{n} = \begin{pmatrix} 1 & 1/2^n \\ 0 & 1 \end{pmatrix} \in H$$

Take any finite collection $\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix}$ in H.

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}^{j_1}\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}^{j_2}\cdots\begin{pmatrix} 1 & x_n \\ 0 & 1 \end{pmatrix}^{j_n} = \begin{pmatrix} 1 & \sum j_i x_i \\ 0 & 1 \end{pmatrix}$$

denominator of every x so that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in H$ is at most product of denoms of x_i, say that's D.

But H contains $\begin{pmatrix} 1 & 1/2^m \\ 0 & 1 \end{pmatrix}$ for arbitrarily large m.