The Euclidean group

Definition: The Euclidean group $E_n(\mathbb{R})$ is the group of distance preserving maps $f : \mathbb{R}^n \to \mathbb{R}^n$; that is, maps such that $\|f(x) - f(y)\| = \|x - y\|$ for all pairs of points x, y. Such a map is called an *isometry*.

$\|f(x)-f(y)\|^2 = \|x-y\|^2$

Definition: Fix a vector a. The map $T(x) = x + a$ is an isometry called a *translation.*

$x \in \mathbb{R}^n$ $a \in \mathbb{R}^n$ $a \neq 0$

$T(x) = x+a$ $T(0) = a$

$\|T(x)-T(y)\|^2$

$= \|x+a-(a+y)\|^2 = \|x-y\|^2.$

Proposition: An isometry that carries the origin to the origin is given by an orthogonal matrix.

Assume $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry $\quad \|f(x)-f(y)\|^2 = \|x-y\|^2$

$f(0) = 0 \quad \to \|f(x)\|^2 = \|x\|^2$

$\|f(x)-f(0)\|^2 = \|x-0\|^2$
$= \|x\|^2.$

① $\langle f(x), f(y)\rangle = \langle x, y\rangle$

$$\langle f(x)-f(y), f(x)-f(y)\rangle = \|f(x)\|^2 - 2\langle f(x), f(y)\rangle + \|f(y)\|^2$$

$$\langle x-y, x-y\rangle = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

so $\langle f(x), f(y)\rangle = \langle x, y\rangle$.

② $e_1, \ldots, e_n$ is an orthonormal basis
$f(e_1), \ldots, f(e_n)$ is too. $\quad \langle f(e_i), f(e_j)\rangle = \langle e_i, e_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$x = \sum x_i e_i$

$$f(x) = \sum_{i=1}^{n} \langle f(x), f(e_i)\rangle f(e_i)$$

$$\langle f(x) - \sum_{i=1}^{n} \langle f(x), f(e_i)\rangle f(e_i), f(e_j)\rangle = 0$$

$$f(x) = \sum_{i=1}^{n} \langle x, e_i\rangle f(e_i) = \sum_{i=1}^{n} x_i f(e_i).$$

$f\left(\sum x_i e_i\right) = \sum x_i f(e_i)$ so f is linear.

f is given by $f(x) = Ax$ for an orthogonal A.

The Euclidean group consists of pairs (A, a) where A is an orthogonal matrix and, if $x \in \mathbb{R}^n$, then

$$(A, a)x = Ax + a.$$

$f(0) = a$

$\tilde{f}(x) = f(x) - a$

$\tilde{f}(x)$ is an isometry that fixes 0.

$\tilde{f}(x) = Ax$

$\tilde{f}(x) = Ax = f(x) - a$

$f(x) = Ax + a.$

- The multiplication $(A, a)(B, b) = (AB, Ab + a)$.

$(B,b)x = Bx + b$

$(A,a)(Bx+b) = A(Bx+b) + a$

$= ABx + Ab + a.$

- The group of translations T consisting of elements $(1, a)$ in $E(n)$ is a normal subgroup isomorphic to $\mathbb{R}^n$.

$(1,a)(1,b) = 1\cdot 1 + 1\cdot b + a = b + a = a + b.$

- The quotient $E(n)/T$ is $O(n)$.

$E(n) \longrightarrow O(n)$

$f: (A,a) \longrightarrow A$

$$\begin{tikzcd} E(n) \arrow[r, "f"] \arrow[d] & O(n) \\ & E(n)/T \arrow[u, equal] \end{tikzcd}$$

$\ker(f) = T$

T is normal

$f((A,a)(B,b)) = f((AB, Ab+a))$

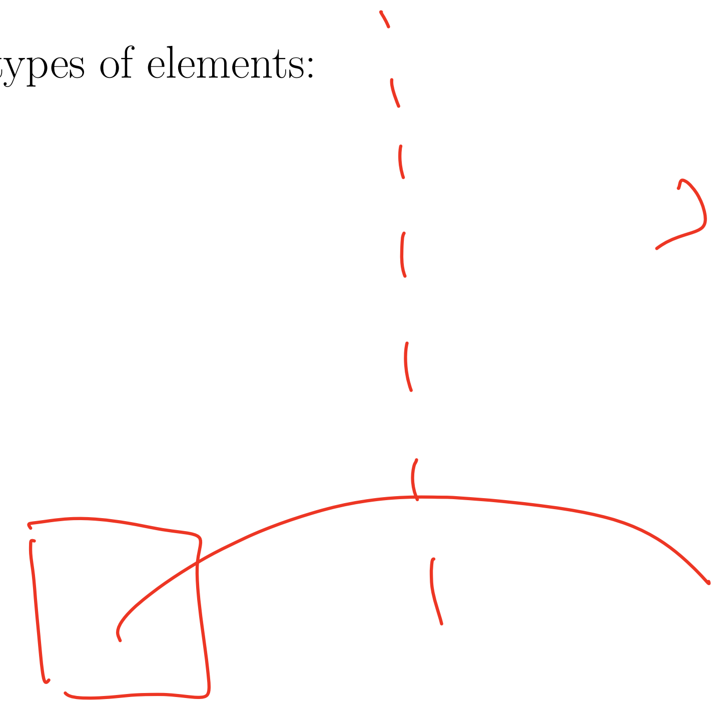
$= AB$

$= f((A,a))f((B,b))$

The plane group

The group $E(2)$ contains four basic types of elements:

- translations
- rotations
- reflections
- glide reflections



Definition: Two regions X, Y in $\mathbb{R}^2$ are congruent if and only if there is a $g \in E(2)$ such that $gX = Y$.

Euclidean geometry = "study of properties invariant by action of $E(2)$".

Finite subgroups of the plane group

Proposition: The only finite subgroups of $E(2)$ are isomorphic to $\mathbb{Z}_n$ or D_n for $n \geq 1$.

(1) $H \leq E(2)$ finite

$E(2)$: $\left.\begin{array}{l}\text{translations} \\ \text{glide } \cancel{\text{translations}} \text{ reflections}\end{array}\right\}$ have infinite order.

reflections

rotations

every $h \in H$ fixes origin.

So $H \leq O(2) \leq E(2)$

$h \in H \quad h = (A, a) \quad a = 0.$

(2) ~~Ex~~ H is a finite subgroup of $O(2)$.

$SO(2) \leq O(2) \qquad SO(2) = \{ R_\theta \}.$

$\left\{ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in [0, 2\pi) \right\}. \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in O(2)$

rotations reflections

hs

$O(2) = SO(2) \cup TSO(2).$

$$TR_\theta T^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_{-\theta}$$

$$(TR_\theta)(TR_\theta) = TR_\theta TR_\theta = T^2 R_{-\theta} R_\theta = 1.$$

$H \leq O(2)$

① $H \leq SO(2)$ has no reflections.

$H = \{R_\theta\}$ $\quad H$ finite.

Among R_θ there must be a smallest angle θ. That $\theta = \frac{2\pi}{n}$.

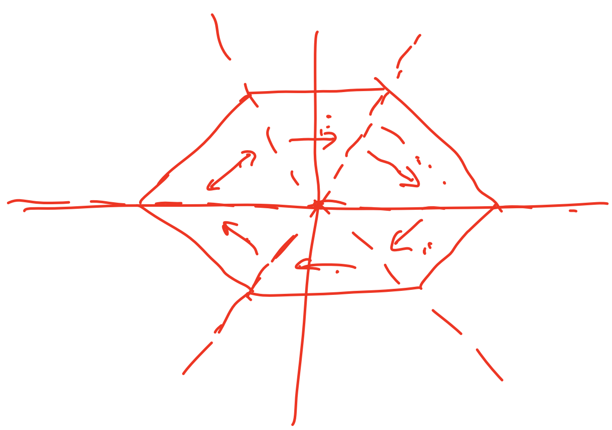
$H \simeq \mathbb{Z}_n$.

$$R_\theta^n = R_{n\cdot\frac{2\pi}{n}} = R_{2\pi} = R_0$$

② H contains a reflection $V = TR_\varphi$

$$H \cap SO(2) = \{1, R_\theta, R_{2\theta}, \dots, R_{(n-1)\theta}\}$$

$$H = \left\{ \begin{array}{l} 1, \dots\dots, R_{(n-1)\theta} \\ V, VR, VR^2, \dots VR^{n-1} \end{array} \right\} \quad \left. \begin{array}{l} V^2 = 1 \\ VRV^{-1} = R^{-1} \\ R^n = e. \end{array} \right\}$$



$n = 6$