Matrix Groups

The general linear group

Definition: The general linear group $\underbrace{\mathrm{GL}_n(\mathbb{R})}$ is the group of bijective linear transformations $T : \underbrace{\mathbb{R}^n \to \mathbb{R}^n}$, with group operation given by composition of maps.

• identity : $I : \mathbb{R}^n \to \mathbb{R}^n$ $\quad I(x) = x$.

• $T : \mathbb{R}^n \to \mathbb{R}^n$ bijective

$T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is in $GL_n(\mathbb{R})$

$T \circ T^{-1} = Id = T^{-1} \circ T$

$U \circ I(x) = U(x)$

$U \circ I = U$

$I \circ U(x) = I(U(x)) = U(x)$

$I \circ U = U$

• $(T \circ U) \circ V\,(x) = (T \circ U)(V(x))$
$= T(U(V(x))) = T \circ (U \circ V)(x)$

Equivalent definition: The general linear group $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible $n \times n$ matrices with matrix multiplication.

$T \longleftrightarrow A$ $\qquad$ T invertible $\Leftrightarrow$ A invertible

$T(x) = Ax$ $\qquad$ $T \circ V \longleftrightarrow AB$

$V \longleftrightarrow B$

$$x = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 2 & 4 \end{pmatrix}$$

Equivalent definition: The general linear group $\mathrm{GL}_n(\mathbb{R})$ is the group of $n \times n$ matrices with nonzero determinant.

$$GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc \neq 0 \right\}.$$

The special linear group

Definition: The special linear group $\mathrm{SL}_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ of matrices with determinant one.

Lemma: $\mathrm{SL}_n(\mathbb{R})$ is normal in $\mathrm{GL}_n(\mathbb{R})$ and the quotient group is $\mathbb{R}^*$.

First isomorphism theorem.

$$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$$

1. $\det(g_1 g_2) = \det(g_1)\det(g_2)$
2. det is surjective $r \neq 0$, $r \in \mathbb{R}$

$$A = \begin{pmatrix} r & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \det(A) = r$$

3. Kernel of det is $SL_n(\mathbb{R})$.

$$\begin{tikzcd} GL_n(\mathbb{R}) \arrow[r, "\det"] \arrow[dr] & \mathbb{R}^* \arrow[d, phantom, "\wr\|"] \\ & GL_n(\mathbb{R})/SL_n(\mathbb{R}) \end{tikzcd}$$

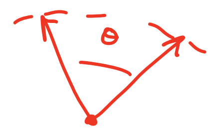
Orthogonal matrices

Definition: A matrix A is called orthogonal if $\|Ax\|^2 = \|x\|^2$ for all vectors $x \in \mathbb{R}^n$.

$A^tA = Id$

Proposition: A is orthogonal if and only if $AA^t = Id$ or, equivalently, if the rows (or columns) of A form an orthonormal set.

Suppose $\|Ax\|^2 = \|x\|^2$ for all $x \in \mathbb{R}^n$.



$\|Ae_i\|^2 = 1 = \|e_i\|^2$ for $i = 1, \dots, n$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ & & & \vdots \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}$$

$\|Ae_i\|^2 = 1$

cols of A are unit vectors.

$$\|A(e_i+e_j)\|^2 = (Ae_i + Ae_j)\cdot(Ae_i + Ae_j)$$

$$= \|Ae_i\|^2 + 2\,\underbrace{(Ae_j \cdot Ae_i)} + \|Ae_j\|^2$$

$$2 = \|e_i + e_j\|^2 = \underbrace{e_i\cdot e_i}_{1} + 2\cancel{e_i\cdot e_j}^{\,0} + \underbrace{e_j\cdot e_j}_{1}$$

$Ae_j \cdot Ae_i = 0$

j^{th} col and i^{th} col of A are orthogonal.

So cols of A are orthonormal.

$$A = \begin{pmatrix} \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$$

$$A^t = \begin{pmatrix} \cdots & a_1 \\ \cdots & a_2 \\ \cdots & \\ \cdots & a_n \end{pmatrix}$$

$$A^tA = \begin{pmatrix} \cdots & a_1 \\ & a_2 \\ & \vdots \\ & a_n \end{pmatrix} \begin{pmatrix} \vdots & \cdots & \vdots \\ a_1 & & a_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{pmatrix} = Id$$

$\|Ax\|^2 = \|x\|^2$ for all x then

$A^t A = id$

$A^t = A^{-1}$

$AA^t = id$

rows of A are orthonormal.

Suppose $AA^t = id$.

$\|Ax\|^2 = Ax \cdot Ax = {}^t(Ax)\, Ax = {}^t x\, {}^t\underbrace{AA}\, x$

$= {}^t x\, Id\, x$

$= {}^t x x$

$= \|x\|^2$

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ ${}^t v = [v_1 \cdots v_n]$

${}^t v v = v \cdot v$

${}^t(AB) = {}^t B\, {}^t A$

The orthogonal group

Definition: The orthogonal group $O_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of orthogonal matrices.

• e orthogonal $e^t e = e$.

• A, B are orthogonal, is AB?

$(AB)^t(AB) = id$?

$B^t \underbrace{A^t A}_{Id} B = \underbrace{B^t B}_{Id} = Id$

$A^{-1}(A^{-1})^t = Id$?

$A^{-1}(A^t)^{-1} = (A^t A)^{-1} = Id$. ~~$AA^t = Id$~~ $A^t = A^{-1}$ $A = (A^{-1})^t = (A^t)^{-1}$

Definition: The special orthogonal group $SO_n(\mathbb{R})$ is the subgroup of $O_n(\mathbb{R})$ consisting of matrices with determinant 1.

Proposition: $SO_n(\mathbb{R})$ is a normal subgroup of $O_n(\mathbb{R})$ of index 2. The quotient group is $\mathbb{Z}_2$.

$\det : O_n \longrightarrow \{\pm 1\}$

$$\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$$

$AA^t = Id$

$(\det(A))^2 = \det(Id) = 1$

$O_n \longrightarrow \{\pm 1\}$

O_n / SO_n

Geometry of the orthogonal group

$SO_n(\mathbb{R})$ is the group of rigid rotations about the origin in $\mathbb{R}^n$.

$SO_2(\mathbb{R})$ is abelian.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$$

$SO_3(\mathbb{R})$ is the group of rotations of the unit sphere.

Frames and orientation

Definition: A frame in $\mathbb{R}^n$ is an ordered orthonormal basis $u_1, \ldots, u_n$.

Frame in $\mathbb{R}^3$ $\qquad \hat{i}, \hat{j}, \hat{k}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\hat{j}, \hat{i}, \hat{k}$

A matrix whose columns are the vectors u_i is orthogonal and so has determinant ± 1.

$$U = \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_n \\ & & & & \\ & & & & \end{bmatrix} \qquad \det U = \pm 1.$$

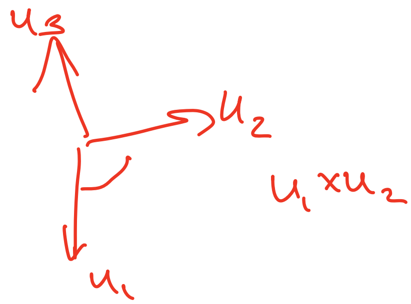
A frame is *positively oriented* if the determinant of this matrix is 1.

$$\begin{array}{ccc} i, & j, & k \end{array} \qquad\qquad \begin{array}{ccc} j, & i, & k \end{array}$$

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \qquad\qquad \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1$$

i, j, k pos oriented

In $\mathbb{R}^3$, positively oriented means "right-handed" so $u_2 \times u_3$ points in the same direction as u_1 where $\times$ is the vector cross product.



$u_1, u_2\ u_3$

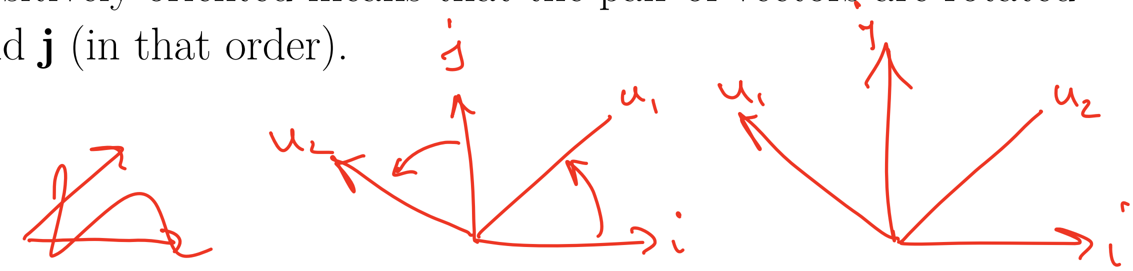
$u_2 \times u_3$ points in u_1 direction

$\Updownarrow$ right handed

$\underbrace{u_1 \cdot (u_2 \times u_3)} = 1$

$\det\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = +1$

In $\mathbb{R}^2$, positively oriented means that the pair of vectors are rotated from **i** and **j** (in that order).



Proposition: SO_n preserves positive oriented frames. If h is in O_n and not SO_n, it changes a positively oriented frame to a negatively oriented one, and vice versa.

$u_1, \ldots, u_n$ orthonormal $\quad A \in SO_n$

$$A \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ & & U & \\ & & & \end{pmatrix} = \begin{pmatrix} Au_1 & & Au_n \\ & & \\ & & \end{pmatrix}$$

$$\det(AU) = \underbrace{\det(A)}_{A \in SO_n}\det(U) = \det(U).$$

Permutations and orthogonal vectors

Let σ be a permutation in S_n. Let T_σ be the linear map that permutes the basis vectors $\mathbf{e}_i$ according to how σ permutes the indices.

$e_1, \ldots\ldots, e_n$ $\qquad (\hat{i}, \hat{j}, \hat{k})$

$\sigma \in S_n$

$\boxed{\sigma e_i = e_{\sigma(i)}}$

$$T_\sigma = \begin{pmatrix} e_{\sigma(1)} & e_{\sigma(2)} & \cdots & e_{\sigma(n)} \\ & & & \end{pmatrix}$$

$\sigma = (12)$

$\sigma(e_1) = e_{\sigma(1)} = e_2$

$\sigma(e_2) = e_{\sigma(2)} = e_1$

$n = 4 \qquad \sigma = (123)$

$\sigma(e_1) = e_2 \qquad \sigma(e_4) = e_4$

$\sigma(e_2) = e_3$

$\sigma(e_3) = e_1$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

M is a perm matrix if it has only zeros and ones with exactly one ~~one~~ in each row and column.

The matrix of T_σ is an orthogonal map.

T_σ is orthogonal.

If σ and σ' are two permutations then $T_{\sigma\sigma'} = T_\sigma T_{\sigma'}$.

$\sigma \to T_\sigma \qquad f : S_n \to O_n$

$\sigma_1 \sigma_2 - T_{\sigma_1\sigma_2} = T_{\sigma_1} \circ T_{\sigma_2}$

$T_{\sigma_1\sigma_2}(e_i) = e_{\sigma_1\sigma_2(i)} =$

$T_{\sigma_1} \circ T_{\sigma_2}(e_i) = T_{\sigma_1}(e_{\sigma_2(i)}) = e_{\sigma_1(\sigma_2(i))} = e_{\sigma_1\sigma_2(i)}$

Proposition: The determinant of T_σ is the sign of σ.

$f: S_n \longrightarrow O_n$

$\det(f(\sigma)) = \pm 1$

Proof: If σ is a transposition

T_σ = identity with two columns switched

and so $\det(T_\sigma) = -1$.

$\sigma = \sigma_1 \cdots \sigma_m$ all σ_i transpositions

$\det(T_\sigma) = \det(T_{\sigma_1 \cdots \sigma_m}) = \det(T_{\sigma_1}) \cdots \det(T_{\sigma_m})$

$= (-1)^n$