Key Properties of Homomorphisms

Proposition: (Proposition 11.4 of the text) Let $\phi : G_1 \to G_2$ be a homomorphism. Then:

- If e_1 is the identity of G_1, $\phi(e_1)$ is the identity of G_2.

Let e_2 be the identity of G_2, e_1 of G_1.

$$e_2\varphi(\cancel{e_1}) \underset{\text{identity in } G_2}{=} \varphi(e_1) \underset{\substack{\cancel{\text{homomorphism}} \\ e_1\cdot e_1 = e_1}}{=} \varphi(e_1\cdot e_1) \underset{\text{homomorphism}}{=} \varphi(e_1)\varphi(\cancel{e_1})$$

$$e_2 = \varphi(e_1)$$

- If g_1 is an element of G_1, then $\phi(g_1^{-1})$ is the inverse $\underbrace{\phi(g_1)^{-1}}$ of $\phi(g_1)$.

$$\underbrace{[\varphi(g_1)]^{-1}}_{\text{in } G_2}\cancel{\varphi(g_1)} = e_2 = \varphi(e_1) = \varphi(g_1^{-1}\cdot g_1)$$
$$= \varphi(g_1^{-1})\cancel{\varphi(g_1)}$$

$$[\varphi(g_1)]^{-1} = \varphi(g_1^{-1})$$

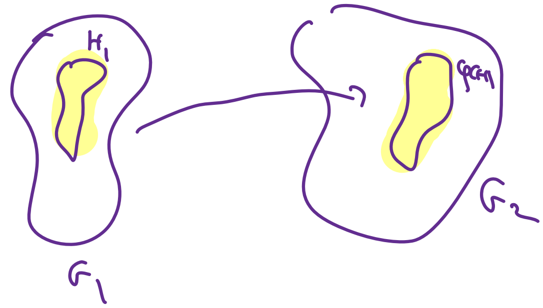
$$G_1 \xrightarrow{\varphi} G_2$$
$$g_1 \longrightarrow \varphi(g_1)$$
$$g_1^{-1} \xrightarrow{\varphi} \varphi(g_1^{-1}) = \varphi(g_1)^{-1}$$

- If H_1 is a subgroup of G_1, then the image $\phi(H_1)$ is a subgroup of G_2.

$$G_1 \xrightarrow{\varphi} G_2$$
$$\cup\!|$$
$$H_1 \qquad \varphi(H_1) \subseteq G_2 \text{ is a subgroup.}$$

$$\varphi(H_1) = \{ g_2 \in G_2 \mid g_2 = \varphi(h_1) \text{ for some } h_1 \in H_1 \}.$$

Must show: $\varphi(H_1)$ not empty and if x, y are in $\varphi(H_1)$ then so is xy^{-1}.



$\varphi(H_1) \neq \emptyset$ because $e_1 \in H_1$

$$e_2 = \varphi(e_1) \in \varphi(H_1)$$

given $x, y \in \varphi(H_1)$

$$x = \varphi(h_1) \qquad \text{for some } h_1, h_2 \in \underline{H_1}$$
$$y = \varphi(h_2)$$

$$xy^{-1} = \varphi(h_1)\underline{\varphi(h_2)^{-1}} = \varphi(h_1)\varphi(h_2^{-1})$$
$$= \varphi(h_1 h_2^{-1})$$
$$h_1 h_2^{-1} \in H_1$$
so $xy^{-1} \in \varphi(H_1)$

- If H_2 is a subgroup of G_2, then the preimage $\phi^{-1}(H_2)$ is a subgroup of G_1.

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \cup\!\!\mid & & \cup\!\!\mid \\ \varphi^{-1}(H_2) & & H_2 \end{array}$$

$$\varphi^{-1}(H_2) = \{ h \in G_1 \mid \varphi(h) \in H_2 \}.$$

- $\varphi^{-1}(H_2) \neq \emptyset$; $e_2 \in H_2$, $\varphi^{-1}(e_2) \in \varphi^{-1}(H_2)$

- ~~chose~~ $x, y \in \varphi^{-1}(H_2)$.

$\cancel{x = \varphi(a)}$ $\quad \varphi(x) = a \qquad a, b \in H_2$

$\varphi(y) = b$

$$\varphi(x)[\varphi(y)]^{-1} = ab^{-1} \in H_2$$

$$\varphi(xy^{-1}) = ab^{-1} \in H_2$$

$$xy^{-1} \in \varphi^{-1}(H_2).$$

- If H_2 is a *normal* subgroup of G_2, then the preimage $\phi^{-1}(H_2)$ is a *normal* subgroup of G_1.

H_2 is normal means

$g_2 H_2 g_2^{-1} = H_2$ for all $g \in G_2$.

Must show

$g_1 \varphi^{-1}(H_2) g_1^{-1} = \varphi^{-1}(H_2)$

for all $g_1 \in G_1$.

Choose $x \in \underline{\varphi^{-1}(H_2)}$

$\varphi(x) = a \in H_2$

$\varphi(g_1 x g_1^{-1}) = \varphi(g_1)\varphi(x)\varphi(g_1^{-1})$

$\varphi(g_1) = g_2 \in G_2$

$g_2 a g_2^{-1} \in H_2$

$\varphi(g_1^{-1}) = g_2^{-1}$

$\varphi(g_1 x g_1^{-1}) \in H_2 \iff \underline{g_1 x g_1^{-1} \in \varphi^{-1}(H_2)}.$

so $\varphi^{-1}(H_2)$ is NORMAL.

Kernels

Definition: The kernel of a homomorphism $\phi : G \to H$ is the preimage of the identity of H:

$$\mathrm{Ker}(\phi) = \phi^{-1}(\{e_H\}). \quad = \{g \in G \mid \varphi(g) = e_H\}.$$

Examples

The kernel of the map $S_n \to \mathbb{Z}_2$ given by $\phi(\sigma) = 0$ if σ is even and $\phi(\sigma) = 1$ if σ is odd is the alternating group A_n.

$$\mathrm{Ker}(\varphi) = \{\sigma \in S_n \mid \varphi(\sigma) = 0 \text{ in } \mathbb{Z}/2\mathbb{Z}\}$$

$$= \underbrace{\{\sigma \mid \sigma \text{ is even}\}}. = A_n$$

$A_n = \mathrm{Ker}(\varphi)$ where φ is the sign homomorphism.

- The kernel of the determinant $\mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ is $\mathrm{SL}_2(\mathbb{R})$.

$g \longrightarrow \det(g)$

$$\mathrm{Ker}(\det) = \{g \mid \det(g) = 1\} \simeq SL_2(\mathbb{R})$$

The kernel of the map $\phi : \mathbb{Z} \to G$ given by $\phi(n) = g^n$ is either $\{0\}$, if g has infinite order, or the subgroup $k\mathbb{Z}$ where k is the order of g in G.

- $\varphi(\mathbb{Z})$ is a subgroup of G.

$$\varphi(\mathbb{Z}) = \{ g^n \mid n \in \mathbb{Z} \} = \langle g \rangle.$$

Kernel of $\varphi := \{ k \in \mathbb{Z} \mid g^k = e \}$.

$g^k = e$ iff order$(g) \mid k$.

or g has infinite order and $k = 0$.

$$\ker(\varphi) = \begin{cases} \{0\} & \text{if } g \text{ has infinite order.} \\ \text{order}(g)\mathbb{Z} & \text{if } g \text{ has finite order} \end{cases}$$

$\{z \mid |z| = 1\}$

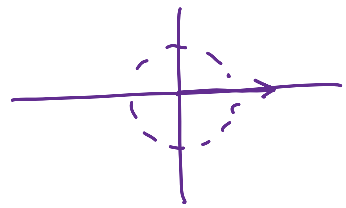
The kernel of the map cis : $\mathbb{R} \to \mathbb{T}$ are the integer multiples of 2π in $\mathbb{R}$.

When is $\operatorname{cis}(r) = 1$? $\qquad$ 1 is the identity in $\mathbb{T}$

$$\operatorname{cis}(r) = \cos(r) + i\sin(r) = 1$$

$$\iff \cos(r) = 1, \quad \sin(r) = 0.$$

$$\Updownarrow$$



$$r = 2\pi n$$

Kernel is $2\pi\mathbb{Z} \leq \mathbb{R}$.

Proposition: (Theorem 11.5 of the text) The kernel of a homomorphism $\phi : G \to H$ is a normal subgroup of G.

- $\mathrm{SL}_2(\mathbb{R})$ is normal in $\mathrm{GL}_2(\mathbb{R})$.
- A_n is normal in S_n.
- Every subgroup of an abelian group is normal.

If $\varphi : G \to H$ is a homomorphism

if $K \subseteq H$ is normal.

then $\varphi^{-1}(K) \subseteq G$ is normal.

$\{e_H\} \subseteq H$ is always normal.

$h\{e_H\}h^{-1} = \{h e_H h^{-1}\} = \{e_H\}$.

$\varphi^{-1}(\{e_H\})$ is always a normal subgroup!