Normal Subgroups: Definition and Examples

Definition: A subgroup H of a group G is *normal* if the left cosets and right cosets of H are the same. That is, H is normal if, for all $g \in G$, $gH = Hg$.

$gH = \{gh : h \in H\}$ $\quad$ $Hg = \{hg : h \in H\}$

Proposition: A subgroup H is normal if and only if

$$gHg^{-1} = \{ghg^{-1} : h \in H\} = H$$

for every $g \in G$.

Proof: Assume $gHg^{-1} = H$. Means that any $h \in H$ can be written $gh'g^{-1}$ $(H \subseteq gHg^{-1})$

Any ghg^{-1} is in H. $(gHg^{-1} \subseteq H)$

Show $gH = Hg$. ① $gH \subseteq Hg$

$x \in gH$ means $x = gh$ for some $h \in H$.

$xg^{-1} = ghg^{-1} \in H$ $\quad$ $xg^{-1} = h'$

$x = h'g$

$x \in Hg$.

Arg. for $Hg \subseteq gH$ is symmetric.

Suppose $gH = Hg$.

Take $h \in H$. $gh = h'g$ for some h'. $ghg^{-1} = h' \in H$.

ghg^{-1} for any $h \in H$ belongs to H. $gHg^{-1} \subseteq H$.

Reverse argument $H \subseteq gHg^{-1}$ is symmetrical.

Basic Examples

- Every subgroup of an abelian group is normal.

Proof: Suppose $H \leq G$ and G abelian.

$ghg^{-1} = h$ for any $h \in H$

since $gh = hg$ $\quad gHg^{-1} = H$ and H is normal.

- the subgroup of rotations of an n-gon is normal in the dihedral group D_n.

D_n: $e, r, r^2, \ldots, r^{n-1}$
$s, sr, \ldots, sr^{n-1}$
$srs = r^{-1}$.

D_n Claim $H = \{e, r, \ldots, r^{n-1}\}$ is normal.

$r^i H = \{r^i, r^{i+1}, \ldots, r^{i+n-1}\} = H$.

$Hr^i = H$

$sr^i H = sH = \{s, sr, \ldots, sr^{n-1}\}$

$Hsr^i = \{s, rs, \ldots, r^{n-1}s\} r^i = \underbrace{\{s, sr^{-1}, \ldots, sr^{1-n}\}}_{} r^i$

$= sHr^i = sH$.

Basic non-Example

- The subgroup of S_3 generated by the transposition (12) is not normal.

S_3 $\quad H = \{e, (12)\}$ $\quad [S_3 : H] = \frac{6}{2} = 3$.

Left H

$\boxed{(123)H = \{(123), (13)\}}$

$(132)H = \{(132), (23)\}$

H $\quad (12)(123) = (23)$

$H(123) = \{(123), (23)\}$

$H(132) = \{(132), (13)\}$

NOT NORMAL!

More Examples

- The group $\mathrm{SL}_2(\mathbb{R})$ of 2×2 matrices with real entries and determinant one is normal in $\mathrm{GL}_2(\mathbb{R})$.

$G = GL_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\}$

$H = SL_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$.

$g \in GL_2(\mathbb{R})$

is $gHg^{-1} \stackrel{?}{=} H$?

$h \in H$, h 2×2 with $\det(h) = 1$.

$\det(ghg^{-1}) = \det(g)\det(h)\det(g^{-1}) = \det(h) = 1.$

So $SL_2(\mathbb{R})$ is normal.

- The subgroup A_n of even permutations is normal in the symmetric group S_n.

S_n symmetric gp of all permutations

$A_n \leq S_n$ is the subset of <u>even</u> permutations.

<u>Pf</u>: Choose $g \in S_n$. Choose $a \in A_n$. Must show $gag^{-1} \in A_n$. in other words we must show that if a is even, so is gag^{-1}.

if g is even: ✓

if g is odd: $g \cdot a$ is odd (odd $\cdot$ even)

$(g \cdot a)g^{-1}$ even (odd $\cdot$ odd)

gag^{-1} is even.

More Examples

- The subgroup $\{-1, 1\}$ is normal in the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$.

2   8

$i^2 = j^2 = k^2 = -1$

$ij = k \quad jk = i \quad ki = j \qquad ij = -ji$

$ik = -ki$

$kj = -jk$

$H = \{-1, 1\}$.

$gHg^{-1} = \underline{H}$.

$i\{-1,1\} = \{-i, i\}$ $\{-1,1\}i = \{-i, i\}$.

$j\{-1,1\} = \{-j, j\} = \{-1,1\}j$

$k\{-1,1\} = \{-1,1\}k$.

- If G is a group, the center $Z(G)$ is normal in G.

G a group

$Z(G)$ is always normal in G.

$Z(G) = \{e, z_1, \ldots, z_k\}$.

for any $g \in G$

$gZ(G) = \{g, gz_1, \ldots gz_k\} = \{g, z_1 g, \ldots, z_k g\}$

$= Z(G)g$.

left and right cosets are the same so $Z(G)$ is normal.

Non-examples

- The subgroup of D_n generated by a reflection s is not a normal subgroup. $n \geq 3$

$H = \{e, s\}$ $|H| = 2$ $|D_n| = 2n$ $[D_n : H] = n.$

Left

$H = \{e, s\}$
$rH = \{r, rs\}$
$r^2H = \{r^2, r^2s\}$
$\vdots$
$r^{n-1}H \quad \{r^{n-1}, r^{n-1}s\}$

n of them

Right

$H = \{e, s\}$
$Hr = \{r, sr\} = \{r, r^{-1}s\}$
$\vdots$
$Hr^{n-1} = \{r^{n-1}, sr^{n-1}\} = \{r^{n-1}, r^{1-n}s\}$

$srs = r^{-1}$

H NOT NORMAL

- The subgroup of S_n ($n \geq 4$) generated by a 3-cycle is not a normal subgroup.

a, b, c all different.!

$H = \{e, (abc), (acb)\}.$ Choose d different from a, b, c.

$r = (ad)$

$(ad)H = \{(ad), (ad)(abc), (ad)(acb)\}$
$= \{(ad), (abcd), (acbd)\}$ left

$H(ad) = \{(ad), (abc)(ad), (acb)(ad)\}$
$= \{(ad), (adbc), (adcb)\}$

NOT NORMAL

- The subgroup H of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices of the form:

$$H = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\}$$

H isomorphic to $\mathbb{R}$.

is not normal.

g $h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$h^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$

Not Normal.

$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad g^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad gg^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{aligned} gHg^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \end{aligned}$$

$$gHg^{-1} = \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{R}\}$$

$gHg^{-1} \neq H$. $\quad$ H NOT NORMAL.