Examples of Factor Groups

- $G = \mathrm{GL}_2(\mathbb{R})$ and $H = \mathrm{SL}_2(\mathbb{R})$. G/H is isomorphic to $\mathbb{R}^\times$.

- $SL_2(\mathbb{R})$ is normal in $GL_2(\mathbb{R})$

To check: if $g \in GL_2(\mathbb{R})$, we must see that $gHg^{-1} = H$.

$h \in SL_2(\mathbb{R})$ $\quad \det(h) = 1$.

$g \in GL_2(\mathbb{R})$ $\quad$ is $ghg^{-1} \in H$?

$$\det(ghg^{-1}) = \cancel{\det(g)}\det(h)\cancel{\det(g^{-1})} = \det(h) = 1$$

So $gHg^{-1} = H$ and H is normal.

- If $g \in G$, if $x \in gH$ then $\det(x) = \det(g)$.

Conversely if $X \in G$ with $\det(x) = \det(g)$

$X(x^{-1}g)$ $\quad xH = x(x^{-1}g)H = gH$.

$\det(x^{-1}g) = 1 \in Sl_2(\mathbb{R})$

$aH = bH \iff \det(a) = \det(b)$.

let $m \in \mathbb{R}^*$. $\quad \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} H = \{ g \in Gl_2(\mathbb{R}) \mid \det(g) = m \}$.

$\det \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} = m$

- We need $f : \mathbb{R}^* \to G/H$.

$$f(m) = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} H.$$

Claim: f is an isomophism.

- f is injective.

if $f(m_1) = f(m_2)$ then $\begin{pmatrix} m_1 & 0 \\ 0 & 1 \end{pmatrix} H = \begin{pmatrix} m_2 & 0 \\ 0 & 1 \end{pmatrix} H$

which means $\det\begin{pmatrix} m_1 & 0 \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} m_2 & 0 \\ 0 & 1 \end{pmatrix}$

$m_1 = m_2$. so f injective.

- f is surjective.

Given gH, let $m = \det(g)$.

$$f(m) = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} H = gH.$$

because $\det\Big(\underbrace{\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}^{-1}}_{m} \underbrace{g}_{m}\Big) = 1$

therefore $f(m)$ is surjective.

- $f(m_1)f(m_2) = f(m_1 m_2)$.

$f(m_1) = m_1 H$

$f(m_2) = m_2 H$

$f(m_1)f(m_2) = (m_1 H)(m_2 H) = m_1 m_2 H = f(m_1 m_2)$.

$$GL_2(\mathbb{R}) / SL_2(\mathbb{R}) \simeq \mathbb{R}^*.$$

- $G = D_n$ where n is even, and $H = Z(G)$. G/H is isomorphic to $D_{n/2}$.

$H = \{e, r^{n/2}\}$

$D_n = \left\{ \begin{matrix} 1, r, r^2, \ldots, r^{n-1} \\ s, sr, sr^2, \ldots sr^{n-1} \end{matrix} \right\}$

$r^n = e$ $\quad \underline{srs = r^{-1}}$

$s^2 = e$

H is normal in D_n.

D_n has $2n$ elements.

D_n/H has n elements

$H, rH, r^2H, \ldots, r^{n/2-1}H,$

$sH, srH, \ldots \ sr^{n/2-1}H$

$n/2 \cdot 2 = n$ cosets

$r^{n/2}H = H$

$r^{n/2}\{e, r^{n/2}\} = \{r^{n/2}, e\} = H$

$rH, (rH)^2 = r^2H, (rH)^3 = r^3H, \ldots (rH)^{n/2-1} = r^{n/2-1}H,$

$(rH)^{n/2} = r^{n/2}H = H.$

rH has order $n/2$.

$(sH)^2 = s^2H = H$

sH has order 2

$(sH)(rH)(sH) = srsH = r^{-1}H$

$D_{n/2} \longrightarrow D_n/Z$

$\sigma^i \longrightarrow r^iH$

$\tau\sigma^i \longrightarrow sr^iH.$

$\left\{ \begin{matrix} 1, \sigma, \sigma^2, \ldots, \sigma^{n/2-1} \\ \tau, \tau\sigma, \ldots, \tau\sigma^{n/2-1} \end{matrix} \right.$

$\boxed{\tau^2 = e, \ \sigma^{n/2} = e \quad \tau\sigma\tau = \sigma^{-1}}$

$D_6/Z(D_6) \cong D_3.$

- $G = Q$ and $H = \{-1, 1\}$. G/H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Q has 8 elts $\pm 1, \pm i \quad \pm j \quad \pm k$

$ij = k$

$ji = -k$

$H = \{-1, 1\}$

$H, \quad \{i, -i\}, \quad \{j, -j\} \quad \{k, -k\}$

$iH \qquad jH \qquad kH$

$\mathbb{Z}_2 \times \mathbb{Z}_2$

$(iH)^2 = -1H = H \qquad -1 \in H$

$(jH)^2 = -1H = H \qquad -1 \in H$

$(kH)^2 = -1H = H \qquad -1 \in H$

$iH \to (1,0)$

$jH \to (0,1)$

$kH = ijH \to (1,1)$

$H \to (0,0)$

$(iH)(jH) = kH$

$(jH)(iH) = -kH = kH$