Factor Groups

Definition: Let G be a group and H be a normal subgroup. Let G/H be the set of left cosets of H in G. Introduce a multiplication on left cosets by the rule

$$(aH)(bH) = abH.$$

$\# G/H = \frac{\# G}{\# H}$ Lagrange's theorem

$G = S_3$

$H = \{e, r, r^2\}$

$G/H = \left\{ \begin{array}{l} H \\ sH \end{array} \right.$ 2 elts

$(H = eH)$

$(eH)(eH) = eH$

$H \cdot H = H$

$sH \cdot sH = s^2 H = eH$

$eH \cdot sH = sH = sH \cdot eH.$

G/H

	eH	sH
eH	eH	sH
sH	sH	eH

$\simeq \mathbb{Z}_2$

Proposition: If G is a group and H is a normal subgroup, the operation on cosets given by $(aH)(bH) = (ab)H$ is well-defined and makes the set

$$G/H$$

of cosets into a group. The identity element is H $= eH$ and the inverse of aH is $a^{-1}H$.

$(aH)(eH) = eH \cdot aH = eaH = aH$

$(aH)(a^{-1}H) = eH$ $\quad a^{-1}H$ looks like inverse.

$H = eH = hH$ for any $h \in H$. $\quad xH = yH \Rightarrow x = yh$

$\Leftrightarrow xH \cap yH \neq \emptyset$

$G = S_3$ $\quad H = \{e, r, r^2\}$

$sH = rsH$ $\quad rs = sr^{-1}$ $\quad r^{-1} \in H$

$sH \cap rsH \supseteq \{rs\}$ $\quad rs \in sH$ $\quad sr^{-1} \in sH$

$\in rsH$

$(sH)(rsH) = srsH$ $\quad srsH = H$

$r^{-1}H = H$

$rsH = sH$

$(sH)(rsH) = sH \cdot sH = s^2H = H$

Well-defined

$\begin{bmatrix} aH = bH \\ cH = dH \end{bmatrix}$

$(aH)(cH) = acH \overset{?}{=} bdH$

$aH = bH$ means $a = bh_1$ $\quad h_1, h_2 \in H$.

$cH = dH$ means $c = dh_2$

$acH = bh_1 d\underbrace{h_2 H}_{H} = bh_1 dH = bh_1(Hd)$

$= b\underbrace{h_1 H}d$

$= b\underbrace{Hd}$

$= bdH$

$Hd = dH$

since H normal

Basic Examples

- $G = \mathbb{Z}$ and $H = n\mathbb{Z}$. $n > 1$

G abelian, H is normal.

cosets of H are $a + n\mathbb{Z}$ $\quad 0 \le a \le n-1$

$a+n\mathbb{Z} \cap a'+n\mathbb{Z} = \emptyset$ if $a \not\equiv a' \bmod n$ [division algorithm]

$a+n\mathbb{Z} = a'+n\mathbb{Z}$ if $a \equiv a' \bmod n$.

So G/H has n cosets $n\mathbb{Z}, 1+n\mathbb{Z}, \ldots, n-1+n\mathbb{Z}$

$$(a+n\mathbb{Z}) + (b+n\mathbb{Z}) = (a+b) + n\mathbb{Z}.$$

$n = 7$

$$(5+n\mathbb{Z}) + (6+n\mathbb{Z}) = 11 + n\mathbb{Z} = 11+7\mathbb{Z} = 4+7\mathbb{Z}$$

$$(5+7\mathbb{Z}) + (6+7\mathbb{Z}) = 4+7\mathbb{Z}.$$

$$(a+n\mathbb{Z}) + (b+n\mathbb{Z}) = (a+b) + n\mathbb{Z}$$

where $a+b \equiv a + b \bmod n$.

$[a] = a+n\mathbb{Z}$ $\qquad [b] = b+n\mathbb{Z}$ $\qquad [a] = [c] \iff a \equiv c \bmod n$

$[b] = [d] \iff b \equiv d \bmod n$

$[c] = c+n\mathbb{Z}$ $\qquad [d] = d+n\mathbb{Z}$

$[a+c] = [b+d]$

if $a \equiv c \bmod n$ and $b \equiv d \bmod n$ then $a+b \equiv c+d \bmod n$.

- $G = \mathbb{Z}_n$ and $H = d\mathbb{Z}_n$ where d is a divisor of n.

$G = \mathbb{Z}_n$ d is a divisor of n. $n \geq 2$

$\langle d \rangle = \{0, d, 2d, \ldots\} \subseteq \mathbb{Z}_n$ $d\mathbb{Z}_n$

what is G/H.

$G = \mathbb{Z}_6$ $d = 2$ $\langle d \rangle = \{0, 2, 4\} = H$

$|G| = 6$ $|\langle d \rangle| = 3$

$\langle d \rangle = H$

$1 + H = 1 + \langle d \rangle = \{1, 3, 5\}$.

$\langle d \rangle \simeq \mathbb{Z}_3 \subseteq \mathbb{Z}_6$

$\mathbb{Z}_6 / \langle d \rangle = \mathbb{Z}_2$.

$G = \mathbb{Z}_n$ $\langle d \rangle = \{0, d, 2d, \ldots, (\frac{n}{d} - 1)d = nd - d\}$

$\langle d \rangle$ has n/d elements

$G/\langle d \rangle$ must have d elements.

$$\left.\begin{array}{l} 0 \; \langle d \rangle \\ 1 + \langle d \rangle \\ \vdots \\ (d-1) + \langle d \rangle \end{array}\right\} d \text{ cosets}$$

they are all different.

$i, j \in \mathbb{Z}_n$

$(i + \langle d \rangle) + (j + \langle d \rangle)$

$= i + j + \langle d \rangle$

$\mathbb{Z}_n / \langle d \rangle = \mathbb{Z}_d$

n/d elts.

$\mathbb{Z}_{20} / \langle 5 \rangle = \mathbb{Z}_5$

$\langle 5 \rangle$ $3 + \langle 5 \rangle$

$1 + \langle 5 \rangle$ $4 + \langle 5 \rangle$

$2 + \langle 5 \rangle$

- $G = D_n$, for $n \geq 3$, and H is the subgroup of rotations.

G has 2n elements   H has n elts.

G/H has 2 elements.

$H = \{e, r, \ldots, r^{n-1}\}$

$sH = \{s, sr, \ldots sr^{n-1}\}$

$H \cdot H = H$, $H \cdot sH = sH \cdot H = sH$; $sH \cdot sH = H$.