Internal direct products

Suppose $G = H \times K$ is a direct product.

- H and K are (isomorphic to) subgroups of G.

$H' = \{(h, e_K) \mid h \in H, \ e_K = \text{identity in } K\}$ $\quad H' \subseteq G$

$K' = \{(e_H, k) \mid k \in K, \ e_H = \text{identity in } H\}.$ $\quad K'$

H': $e_G = (e_H, e_K) \in H'$ $\quad \overset{\in H'}{(h, e_K)} \overset{\in H'}{((h')^{-1}, e_K)} = (h(h')^{-1}, e_K) \in H'$

K' - same argument.

bijective $\to f^{-1}((h, e_K)) = h$. so f has an inverse.

$f: H \longrightarrow H'$

$f(h) = (h, e_K)$ $\quad f((h_1 h_2)) = (h_1 h_2, e_K)$

$= (h_1, e_K)(h_2, e_K) = f(h_1) f(h_2).$

- These copies of H and K commute with one another.

$h' \in H', \ k' \in K'$ $\quad h'k' = k'h'. \ \forall k' \in K' \text{ and } h' \in H'.$

$h' = (h, e_K)$ $\quad h'k' = (h, k)$

$k' = (e_H, k)$ $\quad k'h' = (h, k)$

- The only element these two subgroups have in common is the identity $(1, 1)$. $= (e_H, e_K)$

$H' \cap K' = \{(e_H, e_K)\}$

$H' \ni (h, e_K)$ $\quad (h, e_K) \in K' \Rightarrow h = e_H.$

$g \in G$ $\quad g = (h, k)$

$= \underset{H'}{(h, e_K)} \underset{K'}{(e_H, k)}$

Proposition: Suppose that G is a group and that H and K are two subgroups of G such that:

- $H \cap K = \{e\}$ (1)
- Every $g \in G$ can be written $g = hk$ for some $h \in H$ and some $k \in K$. (This is abbreviated $G = HK$). (2)
- H and K commute, so that $hk = kh$ for all $h \in H$ and $k \in K$. (3)

Then the map

$$f : \underline{H \times K} \to \underline{G}$$

that sends $f(h, k) = hk$ is an isomorphism.

In this case we say that G is the "internal direct product" of H and K.

$f: H \times K \longrightarrow G$ is this an isomorphism?

$f(h,k) = hk$.

is it bijective? - surjective? yes by (2) Every $g = hk$ for some $h \in H$, $k \in K$.

- injective? $f(h,k) = hk = f(h',k') = h'k'$

$hk = h'k'$ then $\underset{H}{(h')^{-1}h} = \underset{K}{k'k^{-1}}$

$(h')^{-1}h \in H \cap K$ so by (1) $(h')^{-1}h = e_H$ or $h = h'$

$(k')k^{-1} = e_K \Rightarrow k = k'$

injective.

$f((h,k)(h',k')) = f(hh', kk') = hh'kk' \overset{?}{=}$

$f((h,k))f((h',k')) = hkh'k'$ By (3) $hkh'k' = hh'kk'$

Example

We know that $\mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$.

Let $H = \{0, 3, 6, 9\}$ be the subgroup generated by 3 and let $K = \{0, 4, 8\}$ be the subgroup generated by 4.

- H is isomorphic to $\mathbb{Z}_4$ and K is isomorphic to $\mathbb{Z}_3$.

$H \cap K = \{0\}$

H and K commute sine G is commutative.

$a \in \mathbb{Z}_{12}$ $\quad a = h + k$

$1 = 4 - 3$

$a = 4 \cdot a + 3(-a)$ for any a.

$11 = 4 \cdot 11 + 3(-11)$ $\quad 4 \cdot 11$ in $\mathbb{Z}_{12}$

$11 \equiv \cancel{8+3}\ 3 + 8$ $\quad 3 \in H$ $8 \in K$.

$44 = 36 + 8 \equiv 8 (12)$

$3 \cdot -11 = -33 \equiv 3 \ (12)$

- G is the internal direct product of H and K.

$G \simeq H \times K$

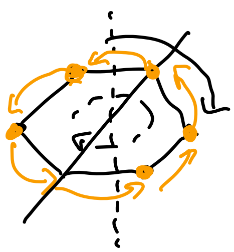
$\mathbb{Z}_{12} \leftarrow \{0, 3, 6, 9\} \times \{0, 4, 8\}$

$x + y \bmod 12 = f(x, y)$

Example

Consider D_6, the symmetries of the regular hexagon. This group is generated by a rotation r and a reflection s, so that r has order 6, s has order 2, and $srs = r^{-1}$.

- Let $H = \boxed{\{e, r^3\}}$ and $K = \boxed{\{e, r^2, r^4, s, r^2s, r^4s\}}$. Then D_6 is the internal direct product of H and K.



r of order 6

s of order 2

$srs = r^{-1}$

$srss = r^{-1}s$

$sr = r^{-1}s$

H: $r^3 \cdot r^3 = e = r^6$

K is a subgroup,

$r^{2i} \cdot r^{2j} = r^{2(i+j)}$

$sr^{2i}s = s\underbrace{r \cdot r \cdot r \cdots r}_{\text{even}} s$

$= r^{-1} \cdot r^{-1} \cdots r^{-1} \cdot \cancel{ss}$

$= r^{-2i}$

- $H \cap K = \{e\}$
- $G = HK$
- $HK = KH$

$r^i s$ — i even $\in K$; i odd

$\underbrace{r^3}_{H \ni}(\underbrace{r^{i-3}s}_{\in K})$

- H is isomorphic to $\mathbb{Z}_2$ and K is isomorphic to S_3, so D_6 is isomorphic to $\mathbb{Z}_2 \times S_3$.

$r^3 \cdot r^{2i} = r^{2i} \cdot r^3$

$r^3 \cdot s = rrrs = sr^{-3} = sr^3$

$r^{-3} = r^3$

$r^6 = 1$

$D_6 = H \times K$

$H \simeq \mathbb{Z}_2$

$K = \{e, r^2, r^4, r^2s, r^4s, s\}$

$\{e, r^2, r^4\}$ is cyclic of order 3,

$sr^2s = r^{-2}$

$K \simeq D_3$ the triangle group.

Example

S_3 is not an internal direct product of non-trivial subgroups.

S_3 has subgroups

$H = \{e, (123), (132)\}$

$K = \{e, (12)\}$,

2 subgroups

$H \cap K = \{e\}$.

$\underline{HK = S_3}$

S_3 is NOT a product.

$(12)e = (12)$

$(12)(123) = (23)$

$(12)(132) = (13)$

$(12)(123) = (23)$ ~~)~~

$(123)(12) = (13)$