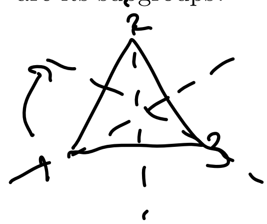
Introduction to Lagrange's Theorem

Theorem: Let G be a finite group and H a subgroup of G. Then the number of elements in H is a divisor of the number of elements in G.

Examples

Let G be the symmetries of the equilateral triangle. G has 6 elements. What are its subgroups?



D_3 = dihedral group $\quad S_3$

e

$(123) \quad (132) \in$

$(23) \quad (13) \quad (12) \in$

	H	$\# H$
a	$\{e\}$	1
b	G	6
c	$\{e, (123), (132)\}$	3
d	$\{e, (12)\}$	2

Suppose H is a subgroup

if $H \neq \{e\}$ then either H contains a rotation or a reflection.

If H contains a rotation then either H is case c or H also contains a reflection.

if H contains $\{e, (123), (132)\}$ and (12) then H also contains

$(123)(12) = (13)$

$(123)(13) = (32)$ so $H = G$.

H contains a reflection (12) then H is case d or it contains a rotation so its all of G or it contains another reflection like (13). Then

$(12)(13) = (321)$ so again $H = G$.

Let G be the group $\mathbb{Z}_{12}$. G has 12 elements. What are its subgroups?

$\langle 1 \rangle = G$ 12 elts

$\langle 2 \rangle = \{2, 4, 6, 8, 10, 0\} \subseteq G$ 6 elts

$\langle 3 \rangle = \{3, 6, 9, 0\} \subseteq G$ 4 elts

$\langle 4 \rangle = \{4, 8, 0\} \subseteq G$ 3 elts

$\langle 6 \rangle = \{6, 0\} \subseteq G$ 2 elts

\# $\langle a \rangle$ is the smallest n so that

$na \equiv 0 \mod 12$.

$$\boxed{\frac{12}{(12, a)} = n}$$

Let G be the symmetric group on 4 elements. What are the orders of different permutations?

S_4 has 24 elts.

(1) has order 1

(21) has order 2

(123) have order 3

(12)(34) have order 2

(1234) have order 4.

order$(g) = \# \langle g \rangle$

By lagrange

$\# \langle g \rangle \mid \# G$.

What about the subgroup A_4?

(1)

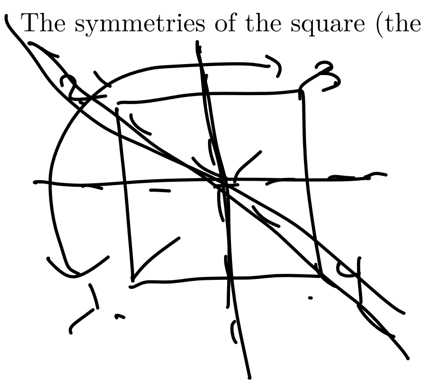
(12)(34) (13)(24) (14)(32)

3 cycles

12

$12 \mid 24$.

The symmetries of the square (the dihedral group D_4) is contained in S_4.



D_4 has 8 elts.

rotations 3 − (1234)

identity − ( )

2 reflections across sides

(23)(14)

2 reflections along digonals.

(13)

$8 \mid 24$