The converse of Lagrange's Theorem is false

Proposition: The group A_4, of order 12, has no subgroup of order 6.

$A_4 = \{ g \in S_4 \mid g \text{ is even} \}$. e, $(12)(34)$, $(13)(24)$, $(14)(23)$

3-cycle.

Proof:

- Suppose H has order 6.

- There are 8 three-cycles in A_4.

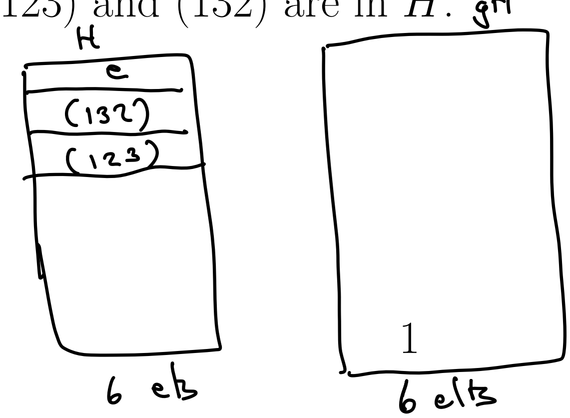
3-cycle: pick 3 of 4 elts $\{1,2,3,4\}$ $\leftrightarrow$ $\binom{4}{3}$ or 4 ways to do that

$\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$

(123), (132)

$4 \times 2 = 8$ ~~3~~cycles, come in pairs g, g^{-1}.

- By the pigeonhole principle, H must contain at least two of them; and since H is a subgroup, it must contain a pair a and a^{-1} where a is some 3-cycle. Suppose for the sake of argument that (123) and (132) are in H.



$[A_4 : H] = 2$

At least 2 3-cycles must be in H.

$(123), (132) \in H$

- There are 3 elements of H unaccounted for. Suppose two of them are three-cycles. Any other three cycle has two elements in common with (123). So suppose (124) and (142) are in H. Then

$$(123)(124) = (13)(24)$$

is in H. That adds up to six elements.

- However also

$$(124)(123) = (23)(14)$$

must be in H, yielding 7 elements, which can't be true.

- That means that the 3 extra elements of H are the (13)(24), (14)(23), and (12)(34). But then

$$(123)(12)(34) = (341)$$

is *also* in H and again we have seven elements.

e
132
123
(13)(24)
(14)(23)
(12)(34)