# Permutations

If $X$ is a set, a bijection $f : X \to X$ is called a *permutation* of $X$

$X$

$f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = x+1$

$X = \{1, 2\}$ $f(1) = 2$ $f(2) = 1$

The permutations of a set form a group $S_X$ under composition of functions, with identity element $id_X : X \to X$.

$id_X(x) = x$

$f \circ (g \circ h)(x) = f(g(h(x))) = (f \circ g) \circ h(x)$

$f$ bijective $\Rightarrow$ $f$ has an inverse function

If $X$ is finite with $n$ elements then $S_X$ has $n!$ elements.

$X =$ $x_1, \ldots, x_n$ $\longrightarrow$ $\left.\begin{array}{l} n \text{ choices for } x_1 \\ n-1 \text{ choices for } x_2 \\ \vdots \\ 1 \text{ choices for } x_n \end{array}\right\} \Rightarrow n \cdot (n-1) \cdot \ldots \cdot 1 = n!$

If $X$ is finite with $n$ elements we can assume

$$X = \{1, 2, \ldots, n\}.$$

The permutations of this set $X$ is called the symmetric group on $n$ elements and written $S_n$. $n!$ elements.

# Multiplication of Permutations

An element $\sigma \in S_n$ sending $\sigma(i) = x_i$ can be written

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

$\{1, \ldots, n\}$

$\sigma(1) = x_1$
$\sigma(2) = x_2$
$\sigma(3) = x_3$
$\vdots$

$\{1, \ldots, n\}$.

Multiplication is *composition of functions* and goes *right to left*.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$\tau(1) = 2$ $\quad$ $\tau(2) = 3$
$\sigma(2) = 1$ $\quad$ $\sigma(3) = 4$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$