8:38 AM Tue Mar 31
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Summary

1. All of the eigenvalues $\lambda_1, \ldots, \lambda_l$ of D are real. If $u^\intercal D u \geq 0$ for all $u \in \mathbf{R}^k$, then all eigenvalues λ_i are non-negative. In the latter case we say that D is *positive semi-definite.*

2. If v is an eigenvector for D with eigenvalue λ, and w is an eigenvector with a different eigenvalue λ', then v and w are orthogonal: $v \cdot w = 0$.

3. There is an orthonormal basis $u_1, \ldots, u_k$ of $\mathbf{R}^k$ made up of eigenvectors of D corresponding to the eigenvalues λ_i.

4. Let Λ be the diagonal matrix with entries $\lambda_1, \ldots, \lambda_N$ and let P be the matrix whose columns are made up of the vectors u_i. Then $D = P\Lambda P^\intercal$.

If we combine our theorem on the critical values with the spectral theorem we get a

$(D = D_0)$ D_0 is a real symmetric matrix

$M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad \det \begin{pmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}$

$= -\lambda(1-\lambda) - (-1)1$

$= \lambda^2 - \lambda + 1 = 0$

$\dfrac{1 \pm \sqrt{-3}}{2}$

$Mx = \lambda x$

$(M - \lambda I)x = 0$

$M - \lambda I$ is not invertible

$\det(M - \lambda I) = 0$ characteristic equation

D covariance matrix

$k \times k$ symmetric

$u^T D u \geq 0$ for all u

Find $u_1, \ldots, u_k$

$\boxed{\|u_i\|^2 = 1 \qquad u_i \cdot u_j = 0 \text{ if } i \neq j}$

$Du_i = \lambda_i u_i$

$P = \begin{pmatrix} u_1 & \cdots & u_k \end{pmatrix}$

$DP = \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \lambda_k u_k \end{pmatrix}$

$= \begin{pmatrix} u_1 & u_2 & \cdots & u_k \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{pmatrix}$

$DP = P\Lambda$

$D = P \Lambda P^{-1}$

$$P P^T = \begin{pmatrix} u_1 & \cdots & u_k \end{pmatrix} \begin{pmatrix} u_1 \cdots \\ u_2 \cdots \\ u_k \cdots \end{pmatrix}$$

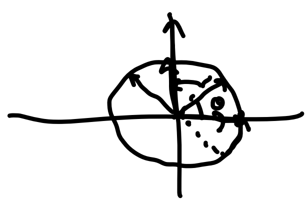
$$= \begin{pmatrix} u_i \cdot u_j \end{pmatrix} \qquad \boxed{P^T = P^{-1}}$$

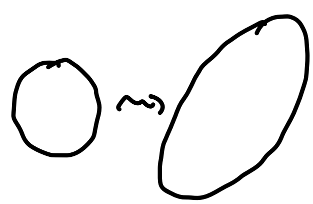
$$D = \underbrace{P \Lambda P^T}_{\text{orthogonal decomposition of } D}$$

any matrix P so that $P^{-1} = P^T$ is called orthogonal

e.g. $P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$P P^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$





D symm matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix}$$

D_0 covariance matrix $K \times K$

spectral theorem gives us

$u_1, \ldots, u_K$

$\|u_i\|^2 = 1, \quad u_i \cdot u_j = 0 \qquad D_0 u_i = \lambda_i u_i$

$$\sigma_{u_i}^2 = u_i^T D_0 u_i = \lambda_i$$

Take our X_0 $\quad$ $\left(D_0 = \frac{1}{N} X_0^T X_0 \right)$

$X_0 u_1, \ldots, X_0 u_k$ $\quad$ X_0 $N \times k$

$\lambda_1, \lambda \ldots \lambda_k$

$X_0 P$ $\quad$ $X_0 = X_0 P P^T$

$N \times k \cdot k \times k$

$N \times k$

$X_0 u_1$