Exam 1 Solutions

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1. (20 points; 10 each) Answer each of the following:

a. Let

$$A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

What is AB?

$$AB = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 10 \\ 4 & 5 \\ 4 & 3 \end{bmatrix}$$

b. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

What is the determininant $\det(A)$?

$$\det(A) = 1(3) + 1(-(3)(-1)) = 3 + 3 = 6$$

2. (5 points each; 20 total) Carefully complete each of the following definitions:

a. A linear combination of a set of vectors $v_1, v_2, \dots, v_k$ in $\mathbf{R}^n$ is: a sum $c_1 v_1 + \cdots c_k v_k$ where $c_1, \dots, c_k$ are constants.

b. The span of a set of vectors $v_1, v_2, \dots, v_k$ in $\mathbf{R}^n$ is: the set of all linear combinations of $v_1, \dots, v_k$.

c. A set of vectors $v_1, v_2, \dots, v_k$ in $\mathbf{R}^n$ is linearly independent if: the only set of constants $c_1, \dots, c_k$ such that $c_1 v_1 + \cdots + c_k v_k = 0$ is the set where all $c_i = 0$.

d. A function $T : \mathbf{R}^m \to \mathbf{R}^n$ is one-to-one if: $T(x) = T(y)$ implies that $x = y$.

3. (20 points) Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ -1 & -1 & 0 \\ 1 & 2 & 3 \\ -2 & -2 & 1 \end{bmatrix}$$

and let

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Explain *briefly* how you would determine if b lies in the span of the columns of A. (For example, what equations would you try to solve, what matrix would you reduce, what would you look for)? You need not actually carry out the solution.

b is in the span of the columns of A if there is a solution to the equation $Ax = b$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. In other words, we need this system of equations to be consistent.

To determine if it is consistent, apply row reduction to the augmented matrix $[A|b]$. The system is consistent if there is no pivot in the last column, or in other words if the last row is zero.

4. (20 points) Applying row reduction to the augmented matrix of a system of 3 linear equations in 4 unknowns resulted in this matrix:

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

Write the solution(s) to the system in parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = u + v$$

where u is a solution of the inhomogeneous system and v is the (family of) solutions to the associated homogeneous system.

The row reduced matrix corresponds to the following system:

$$\begin{aligned} x_1 - 2x_3 &= 1 \\ x_2 + x_3 &= 1 \\ x_3 &= x_3 \\ x_4 &= -3 \end{aligned}$$

So x_3 is the free variable and the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

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5. (20 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be given by $T(x, y, z) = (x - y + z, y + z, x - z)$.

a. (10 points) Find a matrix A so that

$$T(x, y, z) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

b. (5 points) Is T one-to-one? Justify your answer.

We can row reduce A by: - replace row 3 by row 3 - row 1 yielding

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

- replace row 3 by row 3 - row 2 yielding

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0-3 & \end{bmatrix}$$

From this we see that the original matrix is invertible and therefore T is one-to-one by the invertible matrix theorem.

c. (5 points) Is T onto? Justify your answer. Since A is square and invertible T is onto by the invertible matrix theorem.