3.1 Determinants

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Recall the 2x2 case

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix, then the "determinant" of A (written $\det(A)$) is

$$\det(A) = ad - bc$$

The matrix A is invertible if and only if $\det(A) \neq 0$; if it's non-zero then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Generalization

The determinant of an $n \times n$ matrix is defined inductively starting from the 2×2 case.

Let A be an $n \times n$ matrix and let A_i be the $(n-1) \times (n-1)$ submatrix obtained by deleting the i^{th} row and column.

Then (by definition)

$$\det(A) = \sum_{i=1}^{n} (-1)^{n-1} a_{1i} \det(A_i).$$

This works because you can use the same formula on A_i (which is smaller) to find its determinant. When you get to the 2×2 case you know the answer.

3x3 case

$\det(A_1) = a_{22}a_{33} - a_{23}a_{32}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\det(A_2) = a_{21}a_{33} - a_{23}a_{31}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\det(A_3) = a_{21}a_{32} - a_{22}a_{31}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}\det(A_1) - a_{12}\det(A_2) + a_{13}\det(A_3$$

Cofactors

We can generalize the work above by introducing the submatrix A_{ij} obtained by deleting row i and column j from our matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

The i, j *cofactor* C_{ij} of A is $(-1)^{i+j} \det(A_{ij})$. The sign here is important!

Cofactor Expansion

The determinant can be expanded along any row or column yielding the same result.

Fix i and compute along row i:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

or fix j and compute along column j:

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Example (first row expansion)

$$M = \begin{bmatrix} -7 & 4 & 6 \\ 6 & 5 & -7 \\ -2 & -1 & -8 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 5 & -7 \\ -1 & -8 \end{bmatrix}, C_{11} = -47$$

$$M_{12} = \begin{bmatrix} 6 & -7 \\ -2 & -8 \end{bmatrix}, C_{12} = 62$$

$$M_{13} = \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}, C_{13} = 4$$

$$\det(M) = 601$$

Example (second row expansion)

$$M = \begin{bmatrix} -7 & 4 & 6 \\ 6 & 5 & -7 \\ -2 & -1 & -8 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 4 & 6 \\ -1 & -8 \end{bmatrix}, C_{21} = 26$$

$$M_{22} = \begin{bmatrix} -7 & 6 \\ -2 & -8 \end{bmatrix}, C_{22} = 68$$

$$M_{23} = \begin{bmatrix} -7 & 4 \\ -2 & -1 \end{bmatrix}, C_{23} = -15$$

$$\det(M) = 601$$

Triangular Matrices

Suppose that A is an $n \times n$ triangular matrix, meaning that all of the entries a_{ij} where $i > j$ are zero (or all of its entries a_{ij} where $i < j$ are zero)

Then the determinant of A is the product of its diagonal entries.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 7 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = (1)(7)(3) = 21$$

Note: this is a nice result to see by mathematical induction.

Row operations and determinants

The three row operations on a square matrix A have the following effects:

1. Adding a multiple of one row to another does not affect the determinant.
2. Interchanging two rows changes the sign of the determinant.
3. Multiplying a row by a constant k multiplies the determinant by k.

Note: the same is true of “column” operations.

Computing determinants by reduction

$$A = \begin{bmatrix} -5 & 4 & 4 \\ -5 & 2 & -4 \\ 0 & 4 & -4 \end{bmatrix}$$

Replace row 2 with row 2 minus row 1.

$$\begin{bmatrix} -5 & 4 & 4 \\ 0 & -2 & -8 \\ 0 & 4 & -4 \end{bmatrix}$$

Replace row 3 by row 3 plus 2 row 2.

$$\begin{bmatrix} -5 & 4 & 4 \\ 0 & -2 & -8 \\ 0 & 0 & -20 \end{bmatrix}$$

So determinant is $(-5)(-2)(-20) = -200$.

Fundamental properties

Let A and B be $n \times n$ matrices.

1. $\det(AB) = \det(A)\det(B)$.
2. A is invertible if and only if $\det(A)$ is nonzero.
3. $\det(A^T) = \det(A)$

Cramer's Rule

Look at the equation $Ax = b$ where A is an $n \times n$ matrix.

If Y is an $n \times n$ matrix, Let $Y_i(x)$ be the matrix obtained by replacing column i with x and let $Y_i(b)$ be the matrix obtained from Y by replacing the i^{th} column by b.

Then $AI_i(x) = A_i(b)$.

A 2×2 example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} = \begin{bmatrix} a & ax + by \\ c & cx + dy \end{bmatrix} = \begin{bmatrix} a & u \\ c & v \end{bmatrix}$$

so $\det(A)y = \det(A_2(b))$ and therefore

$$y = \det(A_2(b))/\det(A)$$

More on Cramer's Rule

The general form of Cramer's rule is:

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

This is true because $A_i(x) = AI_i(x)$. The determinant of $I_i(x)$ is x_i. So

$$\det(A)x_i = \det(A_i(b))$$

Volumes

Let A be a square matrix of size $n \times n$. The linear map $x \mapsto Ax$ expands volumes by a factor of $|\det(A)|$.

This is a generalization of the fact that the volume of a parallelogram is the base times the height.