2.2-2.3 Matrix Operations

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Elementary Matrices

An elementary matrix is obtained by doing a single row operation on the identity matrix.

There are three types.

Elementary Matrices - Permutations

Suppose

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Switch the first and third rows (for example) and you get

$$I = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Elementary Matrices - Permutations

If E is the elementary matrix obtained from I_n by swapping rows i and j, and A is an $n \times m$ matrix, then EA is obtained from A by swapping rows i and j.

Such an E is an invertible matrix.

Elementary Matrices - scaling

If E is the elementary matrix obtained from I_n by multiplying row i by a, and A is any $n \times m$ matrix, then EA is obtained from A by scaling row i by a.

Such an E is an invertible matrix.

Elementary Matrices - adding

If E is the elementary matrix obtained from I_n by replacing row i by the sum of row i and row j, and A is any $n \times m$ matrix, then EA is the matrix obtained from A by adding rows i and j.

Such an E is an invertible matrix.

Theorem on elementary matrices

If A is an $n \times m$ matrix, there is a sequence of elementary matrices $E_1, \dots, E_k$ so that

$$E_k \cdots E_2 E_1 A$$

is in row reduced echelon from.

If A is a square $n \times n$ matrix, then its row reduced form has only diagonal entries.

Invertible matrices

If the rref of a square matrix A has pivots in every column then A is invertible.

$$E_k \cdots E_2 E_1 A = I_n$$

so

$$A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Invertible matrices

If the rref does *not* have a pivot in every column, then it is not invertible. Because in that case there is a vector v which is not zero such that

$$E_k \cdots E_2 E_1 A v = 0$$

so $Av = 0$. But if A were invertible then $A^{-1}Av = 0$ implies $v = 0$, which is not true.

Computing the inverse

You can compute the inverse matrix of an $n \times n$ square matrix A by finding the RREF of the $n \times 2n$ matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

Theorem on Invertible Matrices

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Figure 1: Inverses

Invertible Linear Maps

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation.

Then T is *invertible* if there is an "inverse function" $S : \mathbf{R}^n \to \mathbf{R}^n$ such that $S(T(x)) = T(S(x)) = x$ for all $x \in \mathbf{R}^n$.

Matrices and maps

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let A be its standard matrix. Then T is invertible if and only if A is an invertible matrix.

If T is invertible, then T is *onto*. Let $x \in \mathbf{R}^n$. Then $T(S(x)) = x$ so $S(x)$ is the element that maps to T. If $y = S(x)$, then $Ay = x$ so the range of A is all of $\mathbf{R}^n$ and so A is invertible.

Conversely if A is invertible then $S(x) = A^{-1}x$ has the desired properties.

Inverses are unique

Note: if T has an inverse, it is one-to-one. Because if $T(x) = T(y)$,then $S(T(x)) = S(T(y))$ and therefore $x = y$.

Note: inverse functions are unique. If $S(T(x)) = T(S(x)) = x$ and $U(T(x)) = T(U(x)) = x$ then $T(S(x)) = T(U(x))$. Since T is one-to-one, $S(x) = U(x)$.

Note: One can show that the inverse of a linear transformation is linear. This is problem 48 on page 149 of the text.

Some T/F problems from the homework

Suppose that A is an $n \times n$ matrix. True or False:

1. If the columns of A span $\mathbf{R}^n$, then they are linearly independent.
2. The equation $Ax = b$ always has at least one solution for each $b \in \mathbf{R}^n$.
3. If the linear transformation $T(x) = Ax$ is one-to-one from $\mathbf{R}^n$ to $\mathbf{R}^n$, then A has n pivots.
4. If A has two identical columns, it is not invertible.
5. If A^T is invertible, so is A.