1.7 Linear Independence

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Linear Independence

A set $v_1, \ldots, v_k$ of vectors in $\mathbf{R}^n$ is *linearly independent* if the only solution to the equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0 \tag{1}$$

is the trivial solution where all $x_i = 0$.

A set of vectors that is not linearly independent is called *linearly dependent*.

A nontrivial solution to equation Equation 1 is called a *linear dependence relation* or just a *linear relation* among the vectors.

Example

Suppose: v1= $\begin{bmatrix} -13 \\ -11 \\ -1 \end{bmatrix}$, v2= $\begin{bmatrix} 7 \\ -14 \\ 8 \end{bmatrix}$, v3= $\begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$

Consider the matrix of the homogeneous system:

$$\begin{bmatrix} -13 & 7 & 1 \\ -11 & -14 & 3 \\ -1 & 8 & -8 \end{bmatrix}$$

Apply row reduction to obtain:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No free variables $\Rightarrow$ no nontrivial solution

Vectors are linearly independent

Example

Suppose: v1= $\begin{bmatrix} -13 \\ -11 \\ -1 \end{bmatrix}$, v2= $\begin{bmatrix} -19 \\ -36 \\ 6 \end{bmatrix}$, v3= $\begin{bmatrix} 44 \\ 97 \\ -19 \end{bmatrix}$

Consider the matrix of the homogeneous system:

$$\begin{bmatrix} -13 & -19 & 44 \\ -11 & -36 & 97 \\ -1 & 6 & -19 \end{bmatrix}$$

Apply row reduction to obtain:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

One free variable so vectors are dependent

Example continued

Recall that v1= $\begin{bmatrix}-13\\-11\\-1\end{bmatrix}$, v2= $\begin{bmatrix}-19\\-36\\6\end{bmatrix}$, v3= $\begin{bmatrix}44\\97\\-19\end{bmatrix}$

The non trivial solution is

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= 3x_3 \end{aligned}$$

so (cancelling out x_3) the linear relation is

$$-v_1 + 3v_2 + v_3 = 0$$

Special cases

- A single vector is a linearly independent set if and only if it is zero.
- Two vectors v_1 and v_2 are linearly independent unless v_2 is a multiple of v_1.
- If $m < n$, then any set of m vectors in $\mathbf{R}^n$ is linearly dependent. This is because a reduced $m \times n$ matrix, which has more columns than rows, must have at least one free variable.

Another characterization

A set $v_1, \dots, v_k$ of vectors in $\mathbf{R}^m$ is dependent if and only if one of the vectors v_i is a linear combination of the others.

- If $v_1 = \sum_{i=2}^{k} c_i v_i$ then

$$-v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

  is the linear relation.

- if $c_1 v_1 + \cdots c_k v_k = 0$ is the linear relation, at least one of the c_i isn't zero, so for that i write

$$-c_i v_i = c_1 v_1 + \cdots + \widehat{c_i v_i} + \cdots + c_k v_k$$

  where $\widehat{c_i v_i}$ means to omit this term. Since $c_i \neq 0$ you can divide this equation by $-c_i$ and write v_i in terms of the other vectors.

Example

For what value(s) of h are the following vectors linearly dependent?

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

They are dependent if v_3 is a linear combination of v_1 and v_2. In other words if

$$v_3 = x_1 v_1 + x_2 v_2$$

has a solution.

$$\begin{array}{ccc} -1 & = & x_1 + 3x_2 \\ 5 & = & -x_1 - 5x_2 \end{array}$$

and then $h = 4x_1 + 7x_2$.

The solution is $x_1 = 5, x_2 = -2$ so $h = 6$ is the only value where the three are dependent.