# Diagonalization of Symmetric Matrices

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# Symmetric Matrices

A matrix $A$ is *symmetric* if $A^T = A$. For example

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & -5 & 7 \\ 1 & 7 & 2 \end{bmatrix}$$

is symmetric.

In our discussion of least squares fitting, we ended up with the matrix $X^T X$ where $X$ was our data matrix. This matrix is symmetric because $(X^T X)^T = X^T X$.

If $U$ is a matrix is an $n \times m$ matrix with columns $u_i$ for $i = 1, \dots, m$, then $U^T U$ is an $m \times m$ matrix whose $i, j$ entry is the dot product of $u_i$ and $u_j$. It is symmetric because the dot product is commutative.

# Eigenvalues/vectors of symmetric matrices

Eigenvalues and vectors of symmetric matrices have special properties. Suppose $A$ is an $n \times n$ symmetric matrix.

Theorem: If $v$ and $w$ are eigenvectors of $A$ with eigenvalues $\lambda$ and $\mu$ where $\lambda \neq \mu$ then $v \cdot w = 0$.

To see this, consider $Av = \lambda v$ and $Aw = \mu w$. Then $w^T A^T = \mu w^T$. But $A^T = A$ wo $w^T A = \mu w^T$.

Remember that $w^T v = w \cdot v$.

This gives us

$$w^T A v = \mu(w^T v) = \mu(w \cdot v) = \lambda(w^T v) = \lambda(w \cdot v)$$

Since $\lambda \neq \mu$ this means $w \cdot v = 0$.

# Orthogonal Diagonalizability

Remember that a square matrix $A$ is diagonalizable if there is a matrix $P$ so that $A = PDP^{-1}$ where $D$ is a diagonal matrix (whose entries are the eigenvalues of $A$).

**Definition:** An $n \times n$ matrix $A$ is *orthogonally diagonalizable* if is an orthonormal set $u_1, \ldots, u_n$ of eigenvectors of $A$.

In other words, $A$ has $n$ mutually orthogonal eigenvectors of length 1.

# Orthogonal Diagonalizability example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Note that $A$ is symmetric. The characteristic polynomial of $A$ is

$$T^2 - 2T - 3 = (T-3)(T+1)$$

so the eigenvalues are $3$ and $-1$.

The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

# Orthogonal diagonalizabiilty example (cont'd)

Notice that $v_1$ and $v_2$ are orthogonal. To make them orthonormal, divide by their norms to get

$$u_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, u_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

We therefore have the equation:

$$A \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

# Orthogonal diagonalizabiilty example (cont'd)

The matrix $P$ is an *orthogonal* matrix:

- its columns are an orthonormal set.
- $P^T = P^{-1}$

# The Spectral Theorem

Symmetric matrices are *special.*

**The Spectral Theorem for Symmetric Matrices**

An $n \times n$ symmetric matrix $A$ has the following properties:

a. $A$ has $n$ real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue $\lambda$ equals the multiplicity of $\lambda$ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. $A$ is orthogonally diagonalizable.

Figure 1: Spectral Theorem

# Spectral Decomposition

Let $u_1, \ldots, u_n$ be the orthonormal eigenvectors of $A$, where $A$ is symmetric. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues (with multiplicity)

Then

$$A = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$$

This writes $A$ as a (weighted) sum of projection operators since $u_i u_i^T v$ gives the projection of $v$ into the $u$ direction.

# Spectral Decomposition example

In our $2 \times 2$ example we had

$$u_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, u_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

So

$$u_1 u_1^T = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

and

$$u_2 u_2^T = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

And

$$A = 3 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$