Gram Schmidt

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Gram Schmidt

In our discussion so far we have been handed orthogonal bases for various subspaces.

How do we find such a thing?

Problem: Given a set of vectors $v_1, \ldots, v_k$ in $\mathbf{R}^n$, find an orthogonal basis (or an orthonormal basis) for the span W of those vectors.

Strategy: Work systematically:

- Start with v_1; it becomes u_1.
- Subtract the component of v_2 in the v_1 direction from v_2; call this u_2.
- Subtract the projection of v_3 into the span of u_1 and u_2 from v_3, and call that u_3.
- Continue in this way, subtracting the projection of v_n from the span of $u_1, \ldots, u_{n-1}$, to obtain u_n.

If you normalize these vectors u_i you get an orthonormal basis.

Gram Schmidt (Example)

Suppose

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The first two vectors in the sequence of G-S vectors is

$$u_1 = v_1, u_2 = v_2 - 3/4v_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

Example (continued)

The third vector

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}$$

Now $u_1 \cdot u_1 = 4$ and

$u_2 \cdot u_2 = v_2 \cdot v_2 - 3/2 v_2 \cdot v_1 + 9/16 v_1 \cdot v_1 = 3 - 9/2 + 9/4 = 3/4$

Also $v_3 \cdot u_2 = v_3 \cdot v_2 - 3/4 v_3 \cdot v_1 = 1/2$. So

$$u_3 = v_3 - \frac{2}{4} u_1 - \frac{2}{3} u_2 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

The QR decomposition

Suppose that A is an $n \times m$ matrix with **linearly independent columns.** Then there is an orthogonal matrix Q (of size $n \times m$) and an upper triangular matrix R of size $m \times m$ so that

$$A = QR$$

The columns of Q form an orthonormal basis for the column space of A; $Q^T Q = I$; and the diagonal entries of R are positive.

(This is called the "QR" decomposition of A).

It's really a restatement of the Gram-Schmidt process.

The QR decomposition

Let A be an $n \times m$ matrix. To compute the QR decomposition, we apply Gram-Schmidt to the columns of A.

Each step in GS corresponds to multiplying A on the right by an upper triangular matrix.

GS and QR example

Suppose that

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -5 & 2 \\ 0 & 2 & -4 \end{bmatrix}$$

We wish to apply Gram-Schmidt to the columns of A. We leave the first column alone. Multiplying A on the right by

$$e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

extracts the first column:

$$Ae_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

GS/QR continued

The next step is to compute

$$u_2 = v_2 - \frac{v2 \cdot u_1}{u_1 \cdot u_1} u_1$$

Since $v_2 \cdot u_1 = 2 - 15 = -13$ and $u_1 \cdot u_1 = 9 + 1 = 10$ this means

$$u_2 = v_2 + \frac{13}{10} u_1 = \begin{bmatrix} 33/10 \\ -11/10 \\ 2 \end{bmatrix}.$$

This second vector can be obtained by multiplying A on the right by

$$e_2 = \begin{bmatrix} 13/10 \\ 1 \\ 0 \end{bmatrix}$$

so that

$$Ae_2 = \begin{bmatrix} 33/10 \\ -11/10 \\ 2 \end{bmatrix}$$

GS and QR continued

Combining steps 1 and 2 we have

$$A \begin{bmatrix} 1 & 13/10 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 33/10 \\ 3 & -11/10 \\ 0 & 2 \end{bmatrix}$$

The last step is to compute

$$u_3 = v_3 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1$$

This gives

$$u_3 = v_3 - \frac{-27/2}{161/10} u_2 - \frac{5}{10} u_1 = \frac{1}{322} \begin{bmatrix} 408 \\ -136 \\ -748 \end{bmatrix}.$$

QR and GS continuedf

In terms of the matrix A, computing u_3 comes from multiplying A on the right by

$$e_3 = \begin{bmatrix} -1/2 \\ 270/322 \\ 1 \end{bmatrix}$$

So we've shown that

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -5 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 13/10 & -1/2 \\ 0 & 1 & 270/32 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 33/10 & 408/322 \\ 3 & -11/10 & -136/322 \\ 0 & 2 & -748/322 \end{bmatrix}$$

QR and GS continued

If we let

$$Q = \begin{bmatrix} 1 & 33/10 & 408/322 \\ 3 & -11/10 & -136/322 \\ 0 & 2 & -748/322 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 13/10 & -1/2 \\ 0 & 1 & 270/32 \\ 0 & 0 & 1 \end{bmatrix}$$

then $Q^T Q = I$ and $AR = Q$.

Also R is invertible (it's diagonal with ones on the diagonal) so $A = QR^{-1}$.

Orthogonal decomposition

The QR decomposition usually has A a square matrix and Q an *orthogonal* matrix meaning that its columns aren't only orthogonal but orthonormal. We can do this by normalizing the columns.

We have

$$u_1 \cdot u_1 = 10, u_2 \cdot u_2 = 161/10, u_3 \cdot u_3 = 2312/322$$

$$Q' = QZ$$

where

$$Z = \begin{bmatrix} 1/\sqrt{10} & 0 & 0 \\ 0 & 1/\sqrt{161/10} & 0 \\ 0 & 0 & 1/\sqrt{2312/322} \end{bmatrix}$$

QR decomposition concluded

Then Q' satisfies $Q^{-1} = Q^T$ and

$$A = Q' Z^{-1} R^{-1} = Q' R'$$

where

$$R' = Z^{-1} R^{-1}$$

is still upper triangular.

Geometric interpretation

A linear transformation like $x \mapsto Ax$ splits into two parts – a shear (coming from the R) and a rotation (coming from Q).