## **Orthogonal Projection**

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### Orthogonal Decomposition

Let W be a subspace of  $\mathbf{R}^n.$  Then every vector  $y\in\mathbf{R}^n$  can be written

$$y = \hat{y} + z$$

where  $\hat{y} \in W$  and  $z \in W^{\perp}$ .

#### Orthogonal decomposition

To compute the decomposition, let  $\{u_1, u_2, \ldots, u_k\}$  be an orthogonal basis of W. Let

$$\hat{y} = \sum_{i=1}^k \frac{y \cdot u_i}{u_i \cdot u_i} u_i.$$

Let  $z = y - \hat{y}$ .

Notice that, for any  $i = 1, \ldots, n$ ,

$$z\cdot u_i = (y-\hat{y})\cdot u_i = y\cdot u_i - \hat{y}\cdot u_i = 0$$

so  $z \in W^{\perp}$ .

The vector  $\hat{y}$  is called the orthogonal projection of y onto W.

# Example

Let

$$y = \begin{bmatrix} -1\\4\\3 \end{bmatrix}$$

and

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Check that  $u_1\cdot u_2=0$  and then find the orthogonal projection of y into the span of  $\{u_1,u_2\}.$ 

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{3}{2}u_1 + \frac{5}{2}u_2$$

SO

$$\hat{y} = \begin{bmatrix} -1\\4\\0 \end{bmatrix}$$

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#### Best approximation

Let W be a subspace of  $\mathbb{R}^n$ . Then  $\hat{y} = \text{proj}_W(y)$  is the point in W that is closest to y among all points in W.

If  $v \in W$ , then

$$|v-y\| \geq \|\hat{y}-y\|$$

for all  $v \in W$ .

$$\|y-v\|^2 = \|(y-\hat{y}) + (\hat{y}-v)\|^2 = \|(y-\hat{y})\|^2 + \|(\hat{(y)}-v)\|^2 + 2(y-\hat{y}) \cdot (\hat{y}-v)$$

The dot product is zero since  $y - \hat{y}$  is in  $W^{\perp}$  and  $\hat{y} - v$  is in W.

Therefore the minimum value occurs when  $\hat{y} - v = 0$ .

#### Distance to a subspace

The distance from a point y to a subspace W is by definition the length of  $y - \hat{y}$  where  $\hat{y}$  is the orthogonal projection onto W.

#### Orthonomal bases

If  $\{u_1,\ldots,u_p\}$  is an orthonormal basis (meaning orthogonal, but all vectors have length one), then we can let

$$U = \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix}$$

be the matrix whose columns are the  $u_i$ .

So U has n rows and p columns.

Then

$$\hat{y} = U U^T y$$

for any  $y \in \mathbf{R}^n$ . This is because  $U^T y$  is the vector whose entries are the  $u_i \cdot y$ .

 $U^T y$  is a vector with p entries, and  $UU^T y$  is the sum of the columns of U – the  $u_i$  weighted by the elements of  $U^T y$ .

#### Examples

Let 
$$u_1 = \begin{bmatrix} 1\\ 3\\ -2 \end{bmatrix}$$
 and  $u_2 = \begin{bmatrix} 5\\ 1\\ 4 \end{bmatrix}$ .

These are orthogonal vectors. Let W be their span. ( W is a plane in  ${\bf R}^3.$  )

Let 
$$y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$
.

These are orthogonal but not orthonormal.

We can make them orthogonal by dividing them by their lengths:

$$v_1 = \frac{1}{\sqrt{14}}u_1$$

and

$$v_2 = \frac{1}{\sqrt{42}}u_2$$

Then we can make the matrix U:

$$U = \begin{bmatrix} 1/\sqrt{14} & 5/\sqrt{42} \\ 3/\sqrt{14} & 1/\sqrt{42} \\ -2/\sqrt{14} & 4/\sqrt{42} \end{bmatrix}$$

SO

The projection of y into W is  $UU^Ty$ .

$$U^{T}y = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{14} & -2/\sqrt{14} \\ 5/\sqrt{42} & 1/\sqrt{42} & 4/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 28/\sqrt{42} \end{bmatrix}$$
$$UU^{T}y = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}$$

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The matrix  $UU^T$  computed directly is

$$\begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}$$

This is a *rank 2* matrix.

That is because its column space is W (which is two dimensional) and its null space is the one dimensional perpendicular to W.

Find the distance from 
$$y = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$$
 to the plane spanned by

$$u_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

The desired distance is

$$\|y-\hat{y}\|$$

where  $\hat{y}$  is the projection of y into the plane spanned by u.

Since

$$\hat{y}=\frac{y\cdot u_1}{u_1\cdot u_1}u_1+\frac{y\cdot u_2}{u_2\cdot u_2}u_2$$

we get

$$\hat{y} = \frac{35}{35}u_1 + \frac{(-28)}{14}u_2 = u_1 - 2u_2 = \begin{bmatrix} 3\\ -9\\ -1 \end{bmatrix}$$

SO

$$\|y - \hat{y}\| = \sqrt{(2)^2 + 0^2 + (6)^2} = \sqrt{40}$$