Orthogonal Projection

Jeremy Teitelbaum

Orthogonal Decomposition

Let W be a subspace of \mathbb{R}^n . Then every vector $y \in \mathbb{R}^n$ can be written

$$
y = \hat{y} + z
$$

where $\hat{y} \in W$ and $z \in W^{\perp}$.

Orthogonal decomposition

To compute the decomposition, let $\{u_1, u_2, ..., u_k\}$ be an orthogonal basis of W . Let

$$
\hat{y} = \sum_{i=1}^{k} \frac{y \cdot u_i}{u_i \cdot u_i} u_i.
$$

Let $z = y - \hat{y}$.

Notice that, for any $i = 1, ..., n$,

$$
z\cdot u_i=(y-\hat y)\cdot u_i=y\cdot u_i-\hat y\cdot u_i=0
$$

so $z \in W^{\perp}$.

The vector \hat{y} is called the orthogonal projection of y onto W .

Example Let

$$
y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}
$$

and

$$
u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
$$

Check that $u_1\cdot u_2=0$ and then find the orthogonal projection of y into the span of $\{u_1,u_2\}.$

$$
\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{3}{2} u_1 + \frac{5}{2} u_2
$$

so

$$
\hat{y} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}
$$

Best approximation

Let W be a subspace of \mathbb{R}^n . Then $\hat{y} = \text{proj}_W(y)$ is the point in W that is closest to y among all points in W .

If $v \in W$, then

$$
\|v-y\|\geq \|\hat{y}-y\|
$$

for all $v \in W$.

$$
\|y-v\|^2=\|(y-\hat y)+(\hat y-v)\|^2=\|(y-\hat y)\|^2+\|(\hat (y)-v)\|^2+2(y-\hat y)\cdot(\hat y-v)
$$

The dot product is zero since $y - \hat{y}$ is in W^{\perp} and $\hat{y} - v$ is in W.

Therefore the minimum value occurs when $\hat{y} - v = 0$.

Distance to a subspace

The distance from a point y to a subspace W is by definition the length of $y - \hat{y}$ where \hat{y} is the orthogonal projection onto W.

Orthonomal bases

If $\{u_1, ..., u_p\}$ is an orthonormal basis (meaning orthogonal, but all vectors have length one), then we can let

$$
U=\begin{bmatrix}u_1 & \cdots & u_p\end{bmatrix}
$$

be the matrix whose columns are the u_i .

So U has n rows and p columns.

Then

$$
\hat{y} = U U^T y
$$

for any $y \in \mathbf{R}^n$. This is because $U^T y$ is the vector whose entries are the $u_i\cdot y$.

 U^Ty is a vector with p entries, and UU^Ty is the sum of the columns of U – the u_i weighted by the elements of $U^T y.$

Examples

Let
$$
u_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}
$$
 and $u_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$.

These are orthogonal vectors. Let W be their span. (W is a plane in ${\bf R}^3$.)

Let
$$
y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}
$$
.

These are orthogonal but not orthonormal.

We can make them orthogonal by dividing them by their lengths:

$$
v_1=\frac{1}{\sqrt{14}}u_1
$$

and

$$
v_2 = \frac{1}{\sqrt{42}} u_2
$$

Then we can make the matrix U :

$$
U = \begin{bmatrix} 1/\sqrt{14} & 5/\sqrt{42} \\ 3/\sqrt{14} & 1/\sqrt{42} \\ -2/\sqrt{14} & 4/\sqrt{42} \end{bmatrix}
$$

so

The projection of y into W is $UU^T y$.

$$
U^{T}y = \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{14} & -2/\sqrt{14} \\ 5/\sqrt{42} & 1/\sqrt{42} & 4/\sqrt{42} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 28/\sqrt{42} \end{bmatrix}
$$

$$
UU^{T}y = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix}
$$

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The matrix UU^T computed directly is

$$
\begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}
$$

This is a *rank 2* matrix.

That is because its column space is W (which is two dimensional) and its null space is the one dimensional perpendicular to W .

Find the distance from
$$
y = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}
$$
 to the plane spanned by

$$
u_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}
$$

The desired distance is

$$
\|y-\hat{y}\|
$$

where \hat{y} is the projection of y into the plane spanned by u .

Since

$$
\hat{y}=\frac{y\cdot u_1}{u_1\cdot u_1}u_1+\frac{y\cdot u_2}{u_2\cdot u_2}u_2
$$

we get

$$
\hat{y} = \frac{35}{35}u_1 + \frac{(-28)}{14}u_2 = u_1 - 2u_2 = \begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}
$$

so

$$
\|y - \hat{y}\| = \sqrt{(2)^2 + 0^2 + (6)^2} = \sqrt{40}
$$