Eigenvalues and Eigenvectors

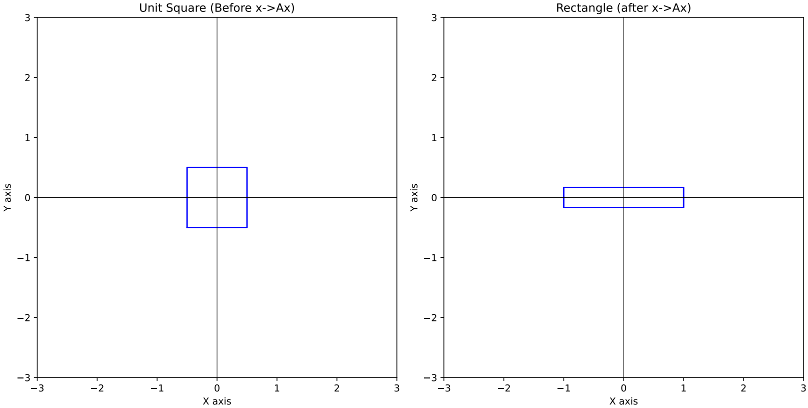
Jeremy Teitelbaum

Eigenvalues and Eigenvectors

If A is a diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

then the linear transformation $x \mapsto Ax$ "stretches" along the x-axis and "shrinks" along the y-axis.

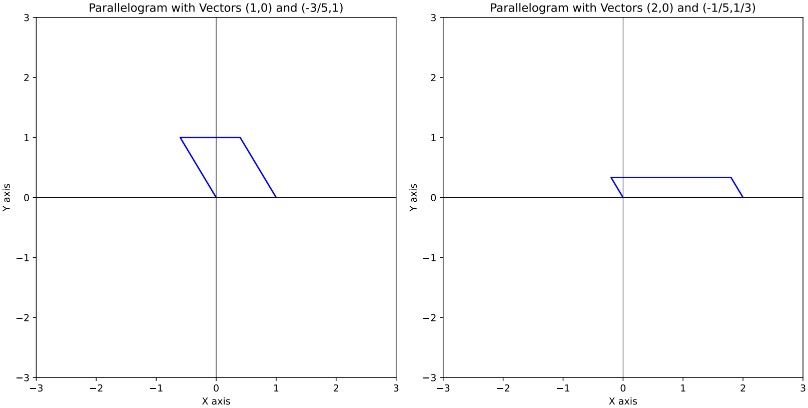


Eigenvalues and Eigenvectors

If A is upper triangular, say

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

then A stretches along the x-axis by 2 as before. Less obviously, it shrinks along the direction given by the vector $(-3/5, 1)$.



Eigenvalues and Eigenvectors

An **eigenvector** for a matrix A is a vector v which gets shrunk or lengthened by A by some factor λ.

The factor λ is called the **eigenvalue**.

More formally, a vector v is called an eigenvector for A (with eigenvalue λ) if v is not zero and

$$Av = \lambda v.$$

Eigenvalues and Eigenvectors

In the example above, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$ are eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

with eigenvalues 2 and $1/3$ respectively.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/3 \end{bmatrix} = (1/3) \begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$$

Triangular Matrices

If A is (upper) triangular then the diagonal entries for A are all eigenvalues. If the diagonal entries are distinct, thenhere are n linearly independent eigenvectors.

Eigenspaces

Suppose that λ is a constant. The vectors v such that

$$Av = \lambda v$$

form a subspace called the *eigenspace* for λ.

This subspace is the nullspace of the matrix

$$A - \lambda I_n$$

where I_n is the $n \times n$ identity matrix.

Independence of Eigenvectors

If $v_1, \dots, v_n$ are eigenvectors for a matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$, and all the λ_i are different, then the v_i are linearly independent. (Note that the v_i are nonzero.)

To see this, suppose that

$$c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0.$$

Then

$$A(c_1 v_1 + c_2 v_2 + \cdots c_n v_n) = c_1 \lambda_1 v_1 + \cdots c_n \lambda_n v_n = 0$$

Multiply the first relation by λ_1 and subtract. You get

$$c_2(\lambda_2 - \lambda_1) v_2 + \cdots + c_n(\lambda_n - \lambda_1) v_n = 0.$$

Since the differences of the λ_i with λ_1 are not zero, we see that $v_2, \dots, v_n$ are dependent.

By repeating this you can show that smaller and smaller collections of the v_i are dependent until you ultimately get $v_n = 0$.

Characteristic Equation

Finding eigenvalues and eigenvectors of a matrix is a hard problem. We can make the following observation.

Suppose λ is an eigenvalue of A where A is an $n \times n$ matrix. Then there is a vector $v \neq 0$ so that $Av = \lambda v$. This means that the matrix $A - \lambda I_n$ is *not* invertible because v is in its null space.

As a result, $\det(A - \lambda I_n = 0$.

Conversely, if $\det(A - \lambda I_n) = 0$, then there is a vector v in the null space and that v is an eigenvector.

It turns out that $\det(A - \lambda I_n)$ is a polynomial in λ, so the eigenvalues of A are the roots of this polynomial.

Example

Let

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

The determinant of $A - \lambda$ is

$$\det(\begin{bmatrix} 3-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix}) = (3-\lambda)(4-\lambda) - 10$$

The polynomial on the right is

$$(3-\lambda)(4-\lambda) - 10 = \lambda^2 - 7\lambda + 12 - 10 = \lambda^2 - 7\lambda + 2.$$

Its roots are $\frac{7 \pm \sqrt{41}}{2}$. These are the eigenvalues of A; they are approximately 6.70156 and 0.29843.

Example continued

To find the eigenvectors, we have to compute the null space of $A - \lambda I$. This is no fun algebraically but with some work you find that the eigenvectors are:

$$\begin{bmatrix} -\frac{\sqrt{41}}{4} - \frac{1}{4} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{41}}{4} \\ 1 \end{bmatrix}$$

Bigger matrices

The characteristic polynomial of an $n \times n$ matrix is of degree n.

Theorem: A polynomial of degree n has n complex roots (counted correctly).

However, in practice one must use numerical methods to find roots of a polynomial of degree 3 or higher.

If one know an eigenvalue λ *exactly* (which usually doesn't happen) then you can find the eigenvectors by computing the null space of $A - \lambda$.

Example

Let

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Since A is (lower) triangular, the eigenvalues are 5,0 and -2. The vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the eigenvector with eigenvalue 5.

Example continued

To find the eigenvector with eigenvalue 0, we must solve

$$Ax = 0.$$

By one of our techniques this is

$$\begin{bmatrix} -8 \\ 5 \\ 0 \end{bmatrix}$$

Example continued

The final eigenvector is in the null space of $A + 2I$:

$$A = \begin{bmatrix} 7 & -8 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

This is

$$\begin{bmatrix} -58/7 \\ -7 \\ 2 \end{bmatrix}$$

Similarity

Two (square) matrices A and B are *similar* if there is an invertible matrix P so that $A = PBP^{-1}$.

Similar matrices have the same eigenvalues because they have the same characteristic polynomial.

$$
\begin{aligned}
\det(PBP^{-1} - \lambda I) &= \det(P(B - \lambda I)P^{-1}) \\
&= \det(P)\det(B - \lambda I)\det(P^{-1}) \\
&= \det(B - \lambda I)
\end{aligned}
$$

Diagonalization

Diagonal matrices are the simplest to work with.

Definition: A square matrix is diagonalizable if there is an invertible matrix P so that $A = PDP^{-1}$ where P diagonal. In other words, A is diagonalizable if it is similar to a diagonal matrix.

Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

In fact, if $A = PDP^{-1}$, then the columns of P are n linearly independent eigenvectors for A, and the diagonal entries of D are the corresponding eigenvalues. This is because in this situation,

$$AP = PD.$$

Example (from the text)

Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

1. The eigenvalues of A. The text tells us that the characteristic polynomial of A is $-(\lambda - 1)(\lambda + 2)^2$ so the eigenvalues are 1 and -2.

Example continued

We need three linearly independent eigenvectors. So we need the null spaces of $A - I$ and $A + 2I$. The book gives us:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

with eigenvalue 1. For the eigenvalue -2:

A+2I=$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$

Reduced form is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example continued

So null space of $A + 2I$ is two dimensional and spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Therefore

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Example continued

We can compute AP:

$$\text{AP}=\begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}=\text{PD}$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So $A = PDP^{-1}$ and A is diagonalizable.

Matrix Powers

One application of diagonalization is that it makes it feasible to understand A^n when A is a square matrix.

If A is diagonalizable, then there is a diagonal matrix D and a matrix P so that

$$A = PDP^{-1}.$$

Then

$$A^m = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^mP^{-1}$$

and

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}$$

Not all matrices are diagonalizable

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue is 1. The null space of $A - I$ is only one dimensional, spanned by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So there's no basis of eigenvectors, so A can't be diagonalized.

n distinct eigenvalues implies diagonalizable

If A has n different eigenvalues, then it has n linearly independent eigenvectors; thus there is a basis of eigenvectors.

Therefore in this case A is diagonalizable.

But as we saw above, you can have repeated eigenvalues and still be diagonalizable.

The diagonalization Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Figure 1: Diagonalization Theorem

Fibonacci Numbers

The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix}.$$

Then

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$$

Fibonacci continued

Continuing we see that

$$A^n = AA^{n-1} = A \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} =$$
$$\begin{bmatrix} F_n & F_{n+1} \\ F_{n-1} + F_n & F_n + F_{n+1} \end{bmatrix} =$$
$$\begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix} \quad (1)$$

So:

$$A^n = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix}$$

Computation of Fibonacci numbers

Let's try to diagonalize the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and use this to compute A^n.

The characteristic polynomial of A:

$$F(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

so $F(\lambda) = \lambda^2 - \lambda - 1$. This polynomial has two roots:

$$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore A is diagonalizable. To compute A^n we need to find P so that

$$A = P \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} P^{-1}$$

Computation of Fibonacci Numbers continued

The columns of the matrix P are the eigenvectors of A. To find these we must solve

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_{\pm} x & \lambda_{\pm} y \end{bmatrix}$$

which translates into the equations

$$\begin{bmatrix} y \\ x + y \end{bmatrix} = \begin{bmatrix} \lambda_{\pm} x \\ \lambda_{\pm} y \end{bmatrix}$$

Eigenvectors are determined only up to scaling so we can set $x = 1$. Then $y = \lambda_{\pm}$. So

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_{+} & \lambda_{-} \end{bmatrix}$$

Fibonacci numbers

Our final result is that

$$\begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = A^n = \frac{1}{\lambda_- - \lambda_+} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \begin{bmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{bmatrix}$$

A little algebra gives:

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

Consequences

Let ϕ be the "Golden ratio" $\frac{1+\sqrt{5}}{2}$.

1. F_n is approximately $\phi^n/\sqrt{5}$.
2. F_n/F_{n-1} converges to ϕ.

0	0.447	0
1	0.724	1
2	1.171	1
3	1.894	2
4	3.065	3
5	4.960	5
6	8.025	8
7	12.985	13
8	21.010	21
9	33.994	34
10	55.004	55
11	88.998	89