Rank and Change of Basis

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Rank

Let A be an $n \times m$ matrix.

The row rank of A is the dimension of the space spanned by the rows of A.

The column rank of A is the dimension of the space spanned by the columns of A.

The nullity of A is the dimension of the null space of A.

Row rank = column rank

Perhaps surprisingly, the row and column ranks are the same.

To see this, put A into row reduced echelon form. The dimension of the column space is the number of linearly independent columns, which is the number of columns containing a pivot.

The rows of A containing a pivot form a basis for the row space of A.

Since the number of pivots is the same whether you look at rows or columns, the ranks are the same.

The Rank Theorem

Let A be an $n \times m$ matrix. Then:

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns}(A) = m$$

This is because:

- the rank of A is the number of pivot columns
- the dimension of the null space is the dimension of the solution space to $Ax = 0$ which is the number of free variables in the row reduced form of A.

These two numbers (pivots plus free variables) add up to the total number of columns.

More on invertible matrices

If A is square of size $n \times n$, then:

- A is invertible if and only if $\operatorname{rank}(A) = n$.
- A is invertible if and only if $\operatorname{nullity}(A) = 0$.

These are restatements of earlier conditions; the first says that the columns of A are linearly independent, the second says that there are no free variables in the rref for A.

A few things to think about

- If V has dimension n, and H is a subspace of V of dimension n, then $H = V$.
- Suppose that A is a 4×7 matrix. Then the rank of A is *at most* 4 and the nullity of A is *at least* 3.
- Suppose that A is a 7×4 matrix. Then the rank of A is at most 4. The nullity is between 0 and 4.

Change of basis

A choice of a basis for a vector space gives a set of coordinates for that vector space.

If we have *two* bases, then we have two sets of coordinates. How are they related?

Suppose $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are both bases of V.

We can write each x_i in terms of the y_j to get a matrix.

Change of basis

$$\begin{aligned} x_1 &= a_{11}y_1 + a_{21}y_2 + \cdots + a_{n1}y_n \\ &\vdots \\ x_n &= a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

If a vector $v = c_1x_1 + \cdots + c_nx_n$ then, written in terms of the y_i we have

$$\begin{aligned} v = (c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n})x_1 + \cdots \\ +(c_1a_{n1} + \cdots + c_na_{nn})x_n \end{aligned}$$

Change of basis continued

The coordinates $[v]_x$ of v in the x basis are computed from the coordinates $[v]_y$ in the y-basis as:

$$[v]_x = A[v]_y$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

NOTE: The columns of A are the coordinates $[x_i]_y$ of the x-basis elements in terms of the y-basis.

Example

If e_1, e_2 are the standard basis for $\mathbf{R}^2$ and y_1, y_2 are the vectors $(1,1)$ and $(-1,1)$ then to convert from the y_1, y_2 basis to the standard basis we should make the matrix A whose columns are the y_1, y_2 in terms of the standard basis.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So if $v = ay_1 + by_2$ then

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$$

and so $v = (a-b)e_1 + (a+b)e_2$.

Example continued

To go backwards, suppose we have $v = ae_1 + be_2$. The inverse of A is

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (a+b)/2 \\ (b-a)/2 \end{bmatrix}$$

To check:

$$(a+b)/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)/2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Notice also that the columns of A^{-1} are the standard basis written in the y_1, y_2 coordinates.

Change of basis theorem

Theorem 15 (page 275 of the text): Let $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be two bases of a vector space V. Then there is a (unique) $n \times n$ matrix $P_{C \leftarrow B}$ such that

$$[x]_C = P_{C \leftarrow B}[x]_B.$$

This matrix is called the "change of basis" or "change of coordinates" matrix from B to C.

The columns of $P_{C \leftarrow B}$ are the C-coordinates of the B-basis vectors. That is, the i^{th} column of $P_{C \leftarrow B}$ is $[b_i]_C$; its entries are $a_1, \dots, a_n$ where

$$b_i = a_1 c_1 + a_2 c_2 + \cdots + a_n c_n.$$

Computing the change of basis matrix

To compute the change of basis matrix $P_{C \leftarrow B}$, suppose we have $b_1, \dots, b_n$ and $c_1, \dots, c_n$ as vectors in $\mathbf{R}^n$.

Let B be the matrix whose columns are the b_i, and C be the matrix whose columns are the c_i.

Note that B is the change of basis matrix from B to the standard basis, and C is the change of basis matrix from C to the standard basis. So C^{-1} is the change of basis matrix from the standard basis to C.

Form the augmented matrix $[C|B]$ and do row reduction. Since C is invertible this will yield $[I|P]$ where I is the identity matrix.

The matrix P is the desired change of basis matrix $P_{C \leftarrow B}$. In fact $P = C^{-1}B$.

Numerical Example

Suppose that $B = \{(1,1),(1,-1)\}$ and $C = \{(2,1),(1,-2)\}$. We want $P_{C \leftarrow B}$. The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix}$$

The row reduced version is:

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

So the change of basis matrix is

$$\begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

Checking the example.

Let $v = (2, 0) = b_1 + b_2$. The coordinates of v in the b-basis are $(1, 1)$. Multiplying:

$$\begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

Now $4/5(2, 1) + 2/5(1, -2) = (2, 0)$ so the coordinates of $(2, 0)$ in the c-basis are in fact $(4/5, 2/5)$.