Dimension

Jeremy Teitelbaum

Dimension

Theorem: Suppose $B = \{b_1, \dots, b_n\}$ is a basis for a vector space V. Then any set $u_1, \dots, u_m$ of $n+1$ or more vectors is linearly dependent.

Proof: Write each u_i in coordinates using b_i. There are n coordinates $[u_i]_B$ for each i, and $m > n$ such coordinate vectors, so these coordinate vectors are linearly dependent.

More explicitly, since B spans, we can write:

$$\begin{array}{rcl} u_1 & = & a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ & \vdots & \\ u_m & = & a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{array}$$

Dimension continued

Our goal is to find a non trivial solution to $\sum_{j=1}^{m} c_j u_j = 0$. The coefficient of b_i in this linear combination is

$$\sum_{j=1}^{m} c_j a_{ji}.$$

Since the b_i are linearly independent, we must have

$$\sum_{j=1}^{m} c_j a_{ji} = 0$$

for each i.

This is a homogeneous linear system of n equations in m unknowns where $m > n$. Thus it must have a nontrivial solution.

Dimension

Theorem: If V has a basis B with n vectors, then every basis of V has n vectors.

Proof: If B' is another basis, it must have n or fewer elements, because if it had more than n it would be linearly dependent. If it had fewer than n, then the original basis B would be dependent. So the only possibility is that B' also has n elements.

Definition: If V has a finite basis, it is called finite dimensional and the dimension of V is the number of elements in a finite basis. Otherwise we say V is infinite dimensional.

Dimension of subspaces

If H is a subspace of a finite dimensional space V, then the dimension of H is at most that of V; if they have the same dimension, then $H = V$.

Also:

- Any linearly independent subset of a vector space H can be extended to a basis. (Assume H finite dimensional)
- Any spanning set contains a basis.

Construction of a basis

Suppose $b_1, \dots, b_k$ are linearly independent in a vector space V. Either they span V, in which case they are already a basis, or they don't span V.

If they don't span V there is a b_{k+1} in V that is not in the span of the b_i for $i \leq k$. Then $b_1, \dots, b_{k+1}$ is still linearly independent.

If V is finite dimensional of dimension n, this process cannot continue indefinitely because once you have n linearly independent vectors you have a basis.

Every (finite) spanning set contains a basis

Suppose B is a (finite) set of vectors that span V. If B is linearly independent, it is already a basis. If not, then one vector in B is dependent on the others, so you can delete it and the remaining vectors still span.

Repeating this process reduces the size of the set of spanning vectors; eventually this has to reach a basis.

Example

Suppose that

$$H = \left\{ \begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix} : s, t \in \mathbf{R} \right\}$$

What is the dimension of this space? Find a basis.

We can rewrite this as

$$H = \{s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} : s, t \in \mathbf{R}\}$$

so H is spanned by the two given vectors.

Since they are linearly independent (look at the last entry) the dimension of H is 2.

Example

The polynomials 1, $2t$, $4t^2 - 2$ and $8t^3 - 12t$ are called the first four Hermite polynomials. They come up in the solution of certain differential equations.

Show that they form a basis for the polynomials of degree 3, so every such polynomial has a unique expansion in terms of the Hermite polynomials.

Example

Relative to the basis $1, t, t^2, t^3$ the Hermite polynomials have coordinate vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

These are four vectors in a four dimensional space, so they are a basis if they are linearly independent. Check this – note that the associated matrix is already upper triangular.