

# Basis and Linear Independence

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# Basis

A set of vectors in  $\mathbf{R}^n$  (or in any vector space  $V$ ) is called a **basis** if

- ▶ it spans  $V$
- ▶ it is linearly independent.

Examples: if  $A$  is an invertible  $n \times n$  matrix, its columns are linearly independent and span  $\mathbf{R}^n$  and therefore are a basis for  $\mathbf{R}^n$ .

The vectors  $1, x, x^2, \dots, x^n$  span the polynomials of degree at most  $n$  and are linearly independent.

The “standard vectors”  $e_i$  for  $i = 1, \dots, n$  are a basis for  $\mathbf{R}^n$ .

## Subspace basis

The vectors  $(1, 3, 2)$  and  $(-1, -1, 0)$  are linearly independent and span a subspace  $H$  of  $\mathbf{R}^3$ .

Therefore they are a basis for  $H$ .

## Every spanning set contains a basis

If a set  $S$  of vectors  $v_1, \dots, v_n$  spans a subspace  $H$ , then a subset of  $S$  is a basis.

**Proof:** If the vectors are linearly independent, they are already a basis.

If they are dependent, then one is a linear combination of the others. Remove that one from  $S$ . The result still spans.

Continue removing dependent vectors until the remaining vectors are independent, and you've found your basis.

## A basis is a minimal spanning set

If  $H$  is a subspace of  $V$ , suppose you have a bunch of vectors in  $H$ .

Too many vectors makes them dependent. Too few means they can't span. If they are a basis, there are enough to span, but not to become dependent.

## Basis for $\text{Nul}(A)$ .

The null space of  $A$  is spanned by the vectors with weights given by the free variables in the row reduced form of  $A$ .

Those vectors are independent and therefore form a basis.

## Basis for $\text{Col}(A)$ .

Given vectors  $v_1, \dots, v_k$ , make an  $m \times k$  matrix with the  $v_i$  as columns.

To find a linear relation among the columns of  $A$ , we need to solve  $Ax = 0$ .

But  $Ax = 0$  if and only if  $EAx = 0$  where  $E$  is an elementary matrix.

Put another way, row reduction doesn't change the  $x$  such that  $Ax = 0$ .

So we can assume  $A$  is in row reduced echelon form.

## More on basis for $\text{Col}(A)$ .

Once  $A$  is in row reduced form, we see that:

- ▶ the columns corresponding to free variables are linear combinations of the pivot columns
- ▶ the pivot columns are linearly independent.



## Basis for $\text{Col}(A)$ .

The **columns of**  $A$  corresponding to the pivot columns in the row reduced version of  $A$  are a basis for the column space. (note that these are *not* the columns of the reduced matrix).

So: a basis for the null space is made up of  $k$  vectors where  $k$  is the number of free variables, and a basis for the column space is made up of  $r$  vectors where  $r$  is the number of pivot columns.

Notice that  $k + r = n$  where  $n$  is the total number of columns of  $A$ .

## Example

Suppose that

$$A = \begin{bmatrix} 2 & 4 & 5 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix}$$

The row reduced form of  $A$  is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since the first three columns are pivot columns, the first three columns of  $A$  span the column space of  $A$ , and the last column satisfies  $c_4 = c_1 + c_2 - c_3$ .

## Example continued

The nullspace of  $A$  is the solution to the homogeneous system, and it is given by the equations

$$x_1 = -x_4$$

$$x_2 = -x_4$$

$$x_3 = x_4$$

so the null space is spanned by

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

# Null Space and Col Space

## Contrast Between Nul $A$ and Col $A$ for an $m \times n$ Matrix $A$

Nul $A$	Col $A$
1. Nul $A$ is a subspace of $\mathbb{R}^n$ .	1. Col $A$ is a subspace of $\mathbb{R}^m$ .
2. Nul $A$ is implicitly defined; that is, you are given only a condition ( $A\mathbf{x} = \mathbf{0}$ ) that vectors in Nul $A$ must satisfy.	2. Col $A$ is explicitly defined; that is, you are told how to build vectors in Col $A$ .
3. It takes time to find vectors in Nul $A$ . Row operations on $[A \ \mathbf{0}]$ are required.	3. It is easy to find vectors in Col $A$ . The columns of $A$ are displayed; others are formed from them.
4. There is no obvious relation between Nul $A$ and the entries in $A$ .	4. There is an obvious relation between Col $A$ and the entries in $A$ , since each column of $A$ is in Col $A$ .
5. A typical vector $\mathbf{v}$ in Nul $A$ has the property that $A\mathbf{v} = \mathbf{0}$ .	5. A typical vector $\mathbf{v}$ in Col $A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$ , it is easy to tell if $\mathbf{v}$ is in Nul $A$ . Just compute $A\mathbf{v}$ .	6. Given a specific vector $\mathbf{v}$ , it may take time to tell if $\mathbf{v}$ is in Col $A$ . Row operations on $[A \ \mathbf{v}]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$ .
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ .

Figure 1: Null Space vs Col Space

## Row space

The *row space* of a matrix is the span of its rows.

Row operations do not change the row space, so one can find a basis for the row space of  $A$  by putting  $A$  in reduced form.

The rows with a pivot (that is, the nonzero rows) form a basis for the row space.

This is because they are clearly linearly independent (and they span by definition).

# Linear Transformations

A linear transformation (or linear map)  $T : V \rightarrow W$ , where  $V$  and  $W$  are vector spaces, is a function that satisfies

$T(u + v) = T(u) + T(v)$  and  $T(cv) = cT(v)$  for all  $u, v \in V$  and  $c \in \mathbf{R}$ .

The *kernel* of a linear transformation is the set of vectors that map to zero:

$$\text{kernel}(T) = \{x \in V : T(x) = 0\}$$

The *range* or *image* of a linear transformation is the set of vectors  $w \in W$  such that there is a  $v \in V$  with  $T(v) = w$ .

## Coordinate systems

**Unique representation:** Suppose that  $B = \{b_1, \dots, b_n\}$  are a basis for a vector space  $V$ . Then any vector  $v$  can be written in exactly one way as a linear combination of the  $b_i$ :

$$v = c_1 b_1 + \dots + c_n b_n$$

The coefficients  $c_1, \dots, c_n$  are called the *coordinates* of  $v$  relative to the basis  $B$ .

The vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the *coordinate vector* for  $v$  relative to  $B$ .

## Coordinates (example)

Suppose that

$$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These form a basis of  $\mathbf{R}^2$ . If

$$v = c_1 e_1 + c_2 e_2$$

then

$$v = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$$

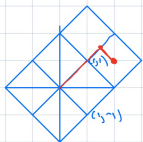


## Coordinates continued

Suppose

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What are the coordinates of  $v$  in the  $e_1, e_2$  basis?



•  $(2,1)$  in "usual  
coords"

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$a + b = 2$$

$$a - b = 1$$

$$a = 3/2$$

$$b = 1/2$$

## Coordinates

In general, each choice of basis for a vector space gives a different system of coordinates on that vector space.

Consider the polynomials with degree at most 2. This vector space has basis  $1, x, x^2$ .

Consider the polynomials  $a(x) = \frac{x(x-1)}{2}$ ,  $b(x) = 1 - x^2$ , and  $c(x) = \frac{x(x+1)}{2}$ .

They form another basis for the degree 2 polynomials.

## Coordinates continued

If

$$f = c_0 + c_1x + c_2x^2$$

then the coordinates of  $f$  in terms of  $a, b, c$  are  $f(-1), f(0), f(1)$ :

$$f(x) = f(-1)a(x) + f(0)b(x) + f(1)c(x).$$

## Coordinates

If  $b_1, \dots, b_n$  is a basis, let  $B$  be the matrix whose columns are the vectors  $b_i$ . Then if we write

$$w = B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

we have  $w = c_1 b_1 + \dots + c_n b_n$  and so the  $c_i$  are the coordinates of  $w$  relative to  $B$ . To *find* the  $c_i$  for a given  $w$ , we need the inverse of  $B$ :

$$B^{-1}w = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

## Coordinates

In our 2-d example, we have

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

so

$$B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

In particular

$$B^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

as above.