1.1-1.2 Systems of Linear Equations

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Linear Equations

A *linear equation* in variables $x_1, \dots, x_n$ with constants $a_1, \dots, a_n$ and b is an equation where the variables all appear to the first power (only).

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

If $n = 2$, the solution set to an equation

$$a_1x_1 + a_2x_2 = b$$

is a line (hence the name linear). In higher dimensions, the solution set is a "hyperplane".

Systems

A system of linear equations is a collection

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \quad \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k \end{aligned}$$

Note the indexing:

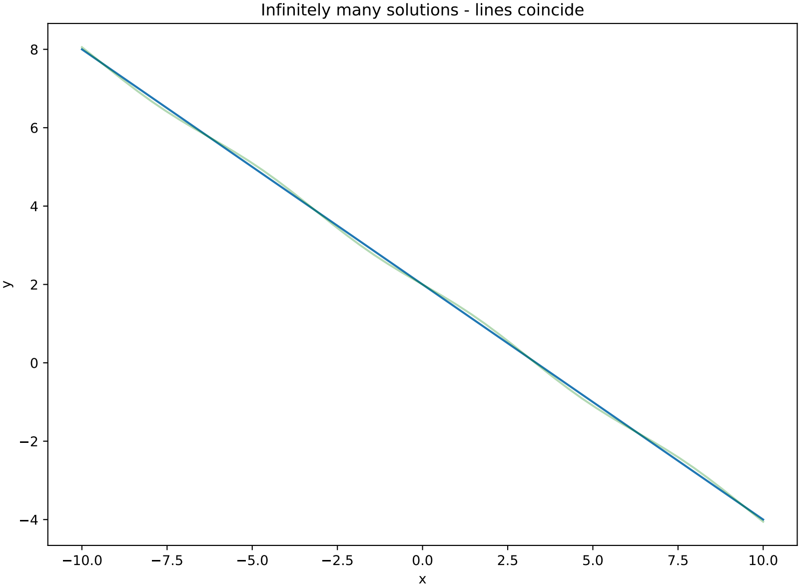
- there are k equations in n unknowns, so there are $k \times n$ coefficients a_{ij}
- there are k constants b_j.

Solutions

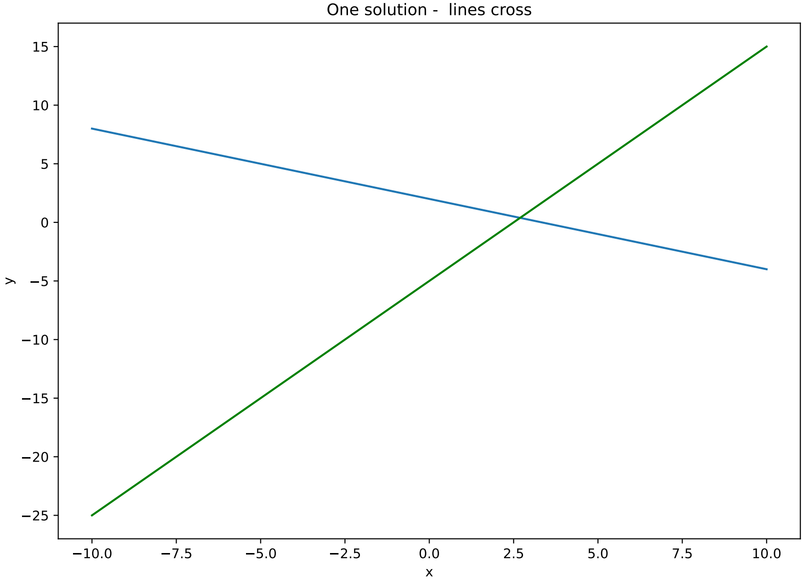
Given two equations in two unknowns there are three possibilities:

- the two equations have infinitely many common solutions.
- the two equations have one common solution.
- the two equations have no common solutions.

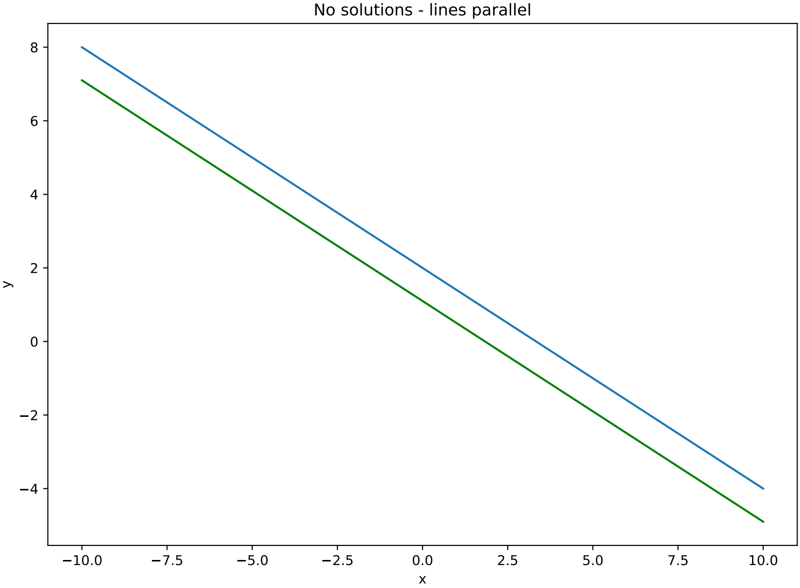
Infinitely many common solutions



One common solution



No common solutions



Goal: Generalize

What can we say about systems with more equations and more unknowns?

Spoiler alert: the same three possibilities hold:

- no solutions
- one solution
- infinitely many solutions

Matrix Equation

We can simplify the writing by replacing this information:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad &\quad \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k \end{aligned}$$

with a "matrix" consisting of just the coefficients.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} & b_k \end{pmatrix}$$

Augmented and Coefficient Matrices

This is called the *augmented matrix* of the system of equations.

If you drop the final "b" column then it's called the coefficient matrix.

Row operations

Given a system of equations, you can:

1. Rearrange the equations into any order.
2. Replace any equation in the system by a multiple by a non-zero constant.
3. Replace any equation e by $e + f$ where f is another equation in the system.

All of these operations are reversible and so the solutions of the transformed system are the same as the original.

These are called *elementary row operations*.

The algorithm for solving a system involves using these row operations to reduce the system to one where the solutions are easy to see.

Row Reduction 1

```
[[ 0.  1.  4. -4.]
 [ 1.  3.  3. -2.]
 [ 3.  7.  5.  6.]]

 Swap row 3 and row 1

[[ 3.  7.  5.  6.]
 [ 1.  3.  3. -2.]
 [ 0.  1.  4. -4.]]

 Row[2] -> -3*Row[2]+Row[1]
```

Row Reduction Continued

```
[[ 3.  7.  5.  6.]
 [ 0. -2. -4. 12.]
 [ 0.  1.  4. -4.]]

 Row[3]-> 2*Row[3]+Row[2]
[[ 3.  7.  5.  6.]
 [ 0. -2. -4. 12.]
 [ 0.  0.  4.  4.]]

Divide rows by leading coeffs

[[ 1.          2.33333333  1.66666667  2.        ]
 [-0.          1.          2.         -6.        ]
 [ 0.          0.          1.          1.        ]]
```

Row Reduction 1 continued

- The reduced matrix tells us that $x_3 = 1$.
- Then from the second row we get: $x_2 + 2x_3 = -6$, so $x_2 + 2 = -6$ and $x_2 = -8$.
- Then from the first row we get $x_1 + (7/3)x_2 + (5/3)x_3 = 2$, so $x_1 - 56/3 + 5/3 = 2$ and $x_1 = 19$.

We should check.

$$x_2 + 4x_3 = -8 + 4 = -4$$

$$x_1 + 3x_2 + 3x_3 = 19 - 24 + 3 = -2$$

$$3x_1 + 7x_2 + 5x_3 = 57 - 56 + 5 = 6$$

Row Reduction 2

```
[[ 0  1 -4  8]
 [ 2 -3  2  1]
 [ 4 -8 12  1]]

Row[2]->-2*Row[2]+Row[3]

[[ 0  1 -4  8]
 [ 0 -2  8 -1]
 [ 4 -8 12  1]]

 Swap Row 3 and Row 1
```

Row Reduction 2 continued

```
[[ 4 -8 12  1]
 [ 0 -2  8 -1]
 [ 0  1 -4  8]]
```

```
Row[3]->2*Row[3]+Row[2]
```

```
[[ 4 -8 12  1]
 [ 0 -2  8 -1]
 [ 0  0  0 15]]
```

This system has no solutions, it is inconsistent - the last row would mean 0=15

Echelon form

A matrix is in *echelon form* (row echelon form) if:

- The zero rows are at the bottom of the matrix
- Each leading non-zero entry in a row is to the right of any leading entry above it.
- The entries below a leading entry are zero.

$$\begin{pmatrix} \square & * & \cdots & * & * & \cdots & * \\ 0 & \square & * \cdots & * & * & \cdots & * \\ 0 & 0 & 0 & \square & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Here $\square$ is non-zero, and $*$ is anything.

Solutions from echelon form

$$\begin{array}{ccccc} x_1 & & +5x_3 & +x_4 & = 11 \\ & 2x_2 & & -x_4 & = 5 \\ & & x_3 & +x_4 & = 1 \end{array}$$

This yields:

$$\begin{aligned} x_3 &= 1 - x_4 \\ x_2 &= 5/2 + x_4/2 \\ x_1 &= 11 - 5(1 - x_4) + x_4 = 6 + 6x_4 \end{aligned}$$

There are infinitely many solutions; x_4 can be anything and the others follow.

Reduced echelon form

A matrix is in reduced echelon form if it is in echelon form and:

- the leading entries are 1
- each leading entry is the only nonzero entry in its column.

Theorem: Given a $k \times n$ matrix, there is a sequence of row operations that will change it into a matrix in reduced row echelon form. A matrix has only *one* reduced row echelon form.

Reduced echelon form continued

Remember our echelon matrix from before

```
[[ 1.          2.33333333  1.66666667  2.        ]
 [-0.          1.          2.         -6.        ]
 [ 0.          0.          1.          1.        ]]
```

We can reduce this

X[1]->X[1]-7/3X[2]

Reduced Echelon Form continued

```
[[ 1.  0. -3. 16.]
 [-0.  1.  2. -6.]
 [ 0.  0.  1.  1.]]

 X[1]->X[1]+3X[3]

[[ 1.  0.  0. 19.]
 [-0.  1.  2. -6.]
 [ 0.  0.  1.  1.]]
X[2]->X[2]-2*X[3]

[[ 1.  0.  0. 19.]
 [-0.  1.  0. -8.]
 [ 0.  0.  1.  1.]]
```

Notice that this “solves” the system explicitly (look at the last column)

Row reduction algorithm (forward pass)

Forward Pass:

1. Find the leftmost column with a nonzero entry. Swap rows to make the top entry in that column nonzero. (This nonzero entry in the top leftmost position is called a pivot).
2. Use row operations to zero out all of the entries below the pivot.
3. Look at the submatrix below the pivot. Carry out steps 1 and 2 on this submatrix. Continue moving down and to the right, applying steps 1 and 2 to smaller and smaller submatrices until you reach the last row.

Row reduction algorithm (backward pass)

4. Now start at the last row which a nonzero entry. Scale that row so its left most nonzero entry is 1.
5. Use row operations to make all the entries in the column above this 1 equal to zero.
6. Now move up and to the left, scaling the leading entry to 1 and eliminating non-zero entries above, until you reach the upper left corner.

Extracting solutions

Let M be the augmented matrix of a linear system. Put M in reduced row echelon form. Then:

0. If there is a row with a non-zero final entry but zeros before that, the system is inconsistent. In other words, if the last column is a pivot column, the system is inconsistent. Otherwise:
1. Columns with a nonzero pivot correspond to *basic variables*.
2. Columns without a pivot correspond to *free variables*.

The free variables can take any value, and the basic variables can be computed for any choice of the free variables.

Classification

- The system has no solutions if the last column of the augmented matrix is a pivot column.
- The system has infinitely many solutions if it is consistent and has at least one free variable.
- The system has a unique solution if every column (except the last one) is a pivot column, and therefore it has no free variables.

Example

```
Matrix is
 1  2  3  4

 4  5  6  7

 6  7  8  9


Reduced matrix is
 1  0  -1  -2

 0  1  2   3

 0  0  0   0
Pivot columns are (1, 2)
Column 3 is a free variable
```

Solutions

$$x_2 = 3 - 2x_3$$
$$x_1 = -2 + x_3$$

Example

```
Matrix is
 1  3  5  7

 3  5  7  9

 5  7  9  1


Reduced matrix is
 1  0  -1  0

 0  1  2   0

 0  0  0   1
Pivot columns are (1, 2)
Column 3 is a free variable

This is an inconsistent system
```