Burnside's Theorem

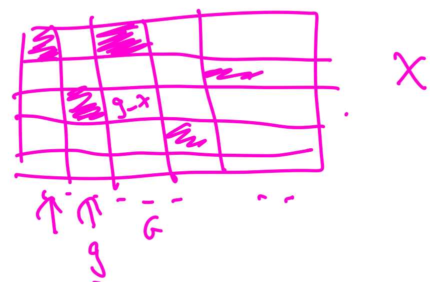
Theorem: Let G be a finite group acting on a set finite X and let k be the number of orbits of X. Then

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

where $X_g = \{x : gx = x\}$ is the fixed point set of g.

Proof:

$G \times X$



$Z \subseteq G \times X$

$Z = \{(g,x) \mid gx = x\}$

Columns

Fix $g \in G$.

How many pairs $(g,x) \in Z$

$(g,x) \in Z$ means $gx = x$

x is a fixed pt for $g \in G$.

$|Z| = \sum_{g \in G} |X_g|$

Rows

Fix $x \in X$

How many $(g,x) \in Z$

$gx = x$

$g \in G_x$.

$|Z| = \sum_{x \in X} |G_x|$

$$\sum_{g \in G} |X_g| = \underline{\sum_{x \in X} |G_x|}$$

$$\sum_{x\in X}|G_x| = \sum_{x\in X}\frac{|G|}{[G:G_x]} = |G|\sum_{x\in X}\frac{1}{[G:G_x]}$$

$$|G_x| = \frac{|G|}{[G:G_x]}$$

$[G:G_x]$ = # of elts in orbit of x.

$$X = \underset{\substack{\uparrow\\ [G:G_{x}] \text{ elts}\\ x\in \text{Orbit}_1}}{\text{Orbit}_1} \cup \underset{\substack{\uparrow\\ [G:G_x]}}{\text{Orbit}_2} \cup \cdots \quad \underset{\substack{\uparrow\\ [G:G_x]\\ x\in \text{orbit } k}}{\text{Orbit}_k}.$$

$$\sum_{x\in X}\frac{1}{[G:G_x]} = \sum_{\text{Orbit}_i}\left(\sum_{y\in \text{Orbit}(i)}\frac{1}{[G:G_y]}\right)$$

$$= \sum_{\text{orbits}}\left(\underbrace{\sum_{\text{elts in orbit}}\frac{1}{\#\text{elts in orbit}}}\right)$$

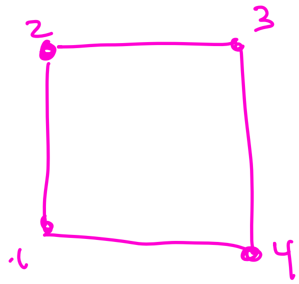
$$= \sum_{\text{orbits}} 1 = k.$$

$$\sum_{x\in X}|G_x| = |G|k.$$

$$\underline{\sum_{g\in G}|X_g| = k\cdot|G|}$$

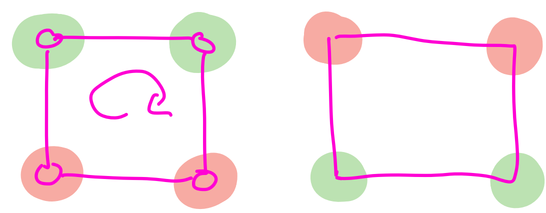
Counting using Burnside's Theorem

Example: How many different ways can you color the vertices of a square using two colors?



Color vertices Red, Green.

16 colorings



$g = R^2$

Coloring:

$f: \{1,2,3,4\} \to \{R, G\}$

16 such functions

$g \in D_4$

$f \circ g = h \iff f$ and h are "the same".

Two colorings are the same if they are in the same orbit for this D_4-action.

How many orbits?

$$K = \frac{1}{|G|} \sum_g |X_g| \qquad |G| = 8$$

$\underline{D_4}$

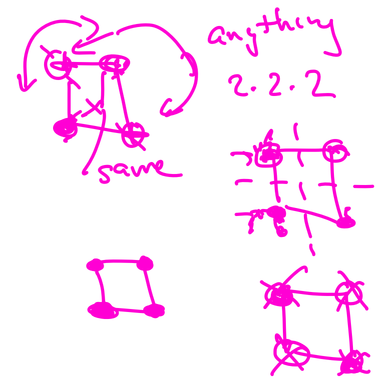
e — 16 colorings fixed by e

2 — 8 colorings fixed here

2 — 4 colorings fixed here

2 — R, R^3 — 2 colorings fixed here

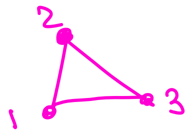
1 — R^2 — 4 colorings fixed here



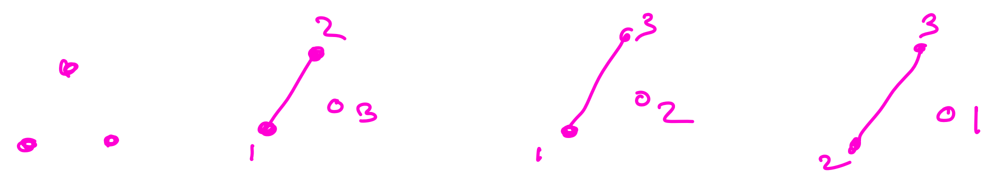
$$\sum |X_g| = 16 + 2\cdot 8 + 2\cdot 4 + 2\cdot 2 + 1\cdot 4 = 16 + 16 + 8 + 4 + 4 = 48$$

$K = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{48}{8} = 6.$

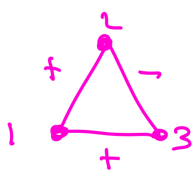
Example: There are 4 different graphs on 3 vertices.



How many graphs are possible with 3 vertices.



2 graphs are equivalent if you can change one to another by renumbering vertices.



$+$ means keep it

$-$ means not there.

Edges $(1,2)$, $(2,3)$, $(1,3)$

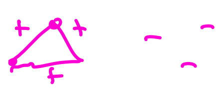
Labels $+, -$

8 labelled graphs

$f : \text{Edges} \to \{+, -\}$

$f \circ \sigma \sim$ 8 h orbits.

8 $|G| = S_3$

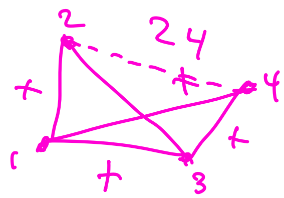
	S_3	Fixed	
1	e		8
3	(12)		$2 \cdot 2 = 4$
2	(123)		2

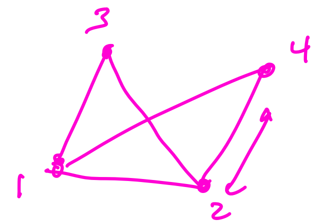
$\sum_g |X_g| = 8 + 3 \cdot 4 + 2 \cdot 2$

$= 8 + 12 + 4 = 24$

$\frac{24}{6} = \underline{4}$ 4 orbits

Example: There are 11 different graphs on 4 vertices.





$\underline{S_4}$

(12) (23) (3,4)
(13) (2,4)
(14)

graph:

(12)	+/−	+
(13)	+/−	+
(14)	+/−	+
(23)	+/−	+
(34)	+/−	+
(2,4)	+/−	−

$(1234) \in S_4$

$(1234)\cdot\underbrace{(1,2)}\longrightarrow\underbrace{(2,3)}$

$(1,4)\longrightarrow(2,1)$

$2^6 = 64$ labellings S_4 acting how many orbits

$K = \frac{1}{|G|}\sum_{g\in G}|X_g|$ X_g = labellings fixed by $g \in G$.

#		$\lvert X_g\rvert$
1	e	64
6	(12)	16
3	(12)(34)	16
8	(123)	4
6	(1234)	4

64	64 ✓
+6·16	96.
+48	48
32	32 ✓
24	24

+/− (12) (13) +/− (14) +/−
(23) (24)
(34) +/−

(12) +/− (13) +/− (14) +/−
(23) (24)
(34) +/−

(12) (13) (14)
+/− (23) (24) +/−
(34)

12 13 14
23 24 +/−
34 +/−

$\frac{1}{24}[96+96+48+24]$
$= 4+4+2+1 = 8+3 = 11$