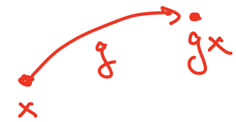
Group Actions

Definition: Let X be a set and G be a group. A (left) action of G on X is a map

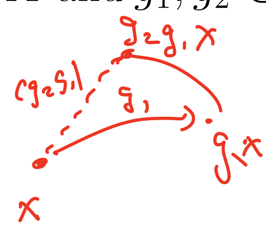
$$G \times X \to X$$
$$(g, x) \mapsto gx$$



such that $ex = x$ and $g_1(g_2x) = (g_1g_2)x$ for all $x \in X$ and $g_1, g_2 \in G$.

$e \cdot x = x$ for all $x \in X$.

$g_2(g_1 x) = (g_2 g_1) x$.



Example 1: Matrix groups acting on $\mathbb{R}^n$.

- $\mathrm{GL}_2(\mathbb{R})$ and its subgroups on $\mathbb{R}^2$.

$g \in GL_2(\mathbb{R})$ $\quad x \in \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \mathbb{R}^2$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(g, x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{g_2}\left(\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g_1}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}\right) = \left(\underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g_1 g_2}\right)\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

- same for $\mathrm{GL}_n(\mathbb{R})$ and its subgroups on $\mathbb{R}^n$.

$H \leq GL_n(\mathbb{R})$

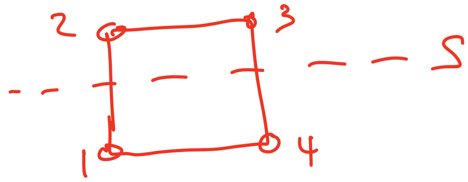
H also acts on $\mathbb{R}^n$ using same map $G \times X \to X$

~~bx~~ but only using $h \in G$.

Example 2: Dihedral groups acting on polygons

- D_n acting on the vertices of the regular polygon with n sides.

D_4



$e \quad R \quad R^2 \quad R^3$

$S, \; SR, \; SR^2, \; SR^3$

$RS = SR^{-1}.$

$R: \; 1 \to 2, \; 2 \to 3, \; 3 \to 4, \; 4 \to 1$

$(1234)(1) = 2.$

$S = 1 \to 2 \quad 2 \to 1 \quad (12)(34)$
$3 \to 4 \quad 4 \to 3$

$S(4) = 3 \qquad S(1) = 2.$

$D_4 \times X \to X$

$X = \{1, 2, 3, 4\}.$

S_n acting on $[1, \ldots, n]$

$\sigma \in S_n$ is a function from $[1, \ldots, n] \to [1, \ldots, n]$

$(\sigma, j) = \sigma(j) \qquad \sigma = (123) \in S_5$

$\sigma \in S_n \quad j \in X = [1, \ldots, n]$

$\sigma(1) = 2 \quad \sigma(3) = 1$

$\sigma(2) = 3 \quad \sigma(\text{else}) = \text{itself}$

Example 3: G **acts on itself by conjugation.**

$X = G$

$G \times X \longrightarrow X$

$(g, x) = gxg^{-1}$

$(e, x) = exe^{-1} = x$

$g_1(g_2 x) = g_1(g_2 x g_2^{-1})$

$= g_1 g_2 x g_2^{-1} g_1^{-1}$

$= (g_1 g_2) x (g_1 g_2)^{-1}$

$= (g_1 g_2) \cdot x$

$\sigma \in S_4$

$\sigma = \underline{(12)(34)}$

$\tau = g\sigma g^{-1} \iff \tau$ has same decomposition

$g \in S_4$

$\tau = \left.\begin{array}{l}(13)(24)\\(14)(23)\\(12)(34)\end{array}\right\}$

Example 4: G acts on the left cosets of a subgroup H.

- Let H be one of the two element subgroups of S_3. Consider the action of S_3 on these cosets.

$G \qquad H \subseteq G.$

G act on left cosets of H by

$(g, xH) = gxH \qquad G \times \text{cosets} \to \text{cosets}.$

$(e, xH) = exH = xH$

$g_1 \cdot (\underbrace{g_2 \cdot xH}_{g_2 \cdot xH}) = g_1(g_2 xH) = g_1 g_2 xH$

$= (g_1 g_2) xH.$

$(g_2, xH) \to g_2 xH.$

$G = S_3 \qquad H = \{e, (12)\}, \quad (13)H = \{(13), (13)(12)\} = \{(13), (123)\}, \quad (23)H = \{(23), (23)(12)\} = \{(23), (132)\}.$

$(123) \cdot H = (123)H = (13)H$

$(123)(13)H = (23)H$

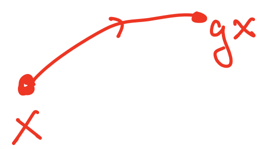
$(123)(23)H = H$

Orbits and stabilizers

Definition: Two points $x, y \in X$ are G-equivalent if there is a $g \in G$ such that $y = gx$. G-equivalence is an equivalence relation and the classes are called **orbits**. Our book writes O_x for the orbit containing x but I like to write Gx.

Example: S_3 acting on itself by conjugation.

G acts on X



- $x \sim x$? yes $x = e \cdot x$.
- if $x \sim y$ is $y \sim x$? $x \sim y$ means $y = g \cdot x$.

$$g^{-1} y = g^{-1} g x = e \cdot x = x$$

- $x \sim y$ and $y \sim z \Rightarrow y = g_1 x \quad z = g_2 y$.

$$z = g_2 g_1 x$$

X gets divided up into classes. ORBITS

Each orbit Gx $\quad O_x$.

S_3 2 permutations are conjugate $\Leftrightarrow$ they have same cycle decomp.

S_3 acting on $X = S_3$

$\sigma = g \tau g^{-1} \Leftrightarrow$ same cycle decomp.

Orbits:

$e \qquad g \cdot e = geg^{-1} = e$.

$(12) \qquad g(12)g^{-1} = (ab)$

$(123) \qquad g(123)g^{-1} = (abc)$.

Definition: If $x \in X$, the set of g such that $gx = x$ is called *the stabilizer subgroup* or just *the stabilizer* of x. It is a subgroup of G written G_x.

Example: S_3 acting on itself by conjugation.

$x \in X$ $\quad G_x = \{g \mid gx = x\}$.

$e \in S_3$ $\quad$ conjugation.

$g \in G_e \Leftrightarrow geg^{-1} = e$ $\quad$ that's true for all g! $G_e = G$.

$g \in S_3$ so that $g(12)g^{-1} = (12)$ $\quad \underline{g = e \quad g = (12)}$ $\quad \{e, (12)\} = G_{(12)}$

$g(12) = (12)g$

$(13)(12)(13) = (23)$ doesn't fix (12)

$(23)(12)(23) = (13)$

~~$g = (123)$~~ $\quad x = (123)$

gxg^{-1} $\quad \{e, (123), (132)\}$ $\quad G_{(123)} = \{e, (123), (132)\}$.

Lemma: G_x is a subgroup.

$G_x = \{g \mid gx = x\}$. $\quad e \in G_x$? yes $ex = x$.

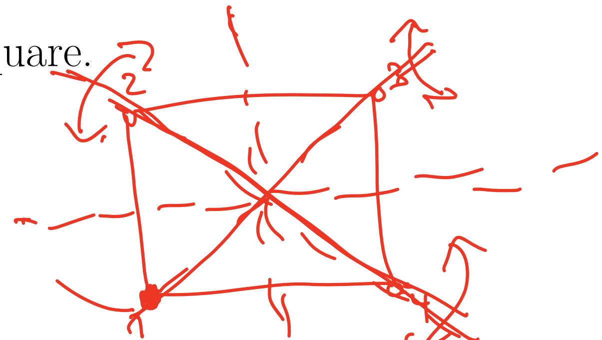
$g_1, g_2 \in G_x$ $\quad g_1(g_2x) = g_1 x = x$

$= (\underline{g_1g_2})x$

$gx = x$ $\quad \underline{g^{-1}gx} = \underline{g^{-1}x}$

$= x$

- D_4 acting on the square.



$\{1,2,3,4\}$ partitioned into orbits.

$R \in D_4$ $\quad R(1)=2 \quad R(2)=3 \quad R(3)=4$

$R(4)=1$

$2 \sim 1$
$3 \sim 2$
$4 \sim 3$
$1 \sim 4$

all vertices are G-equivalent.

There is one orbit $\{1,2,3,4\}$.

$G_{(1)} = \{e, (24)\}$ $\quad G_2 = \{e, (13)\}$

$G_{(3)} = \{e, (24)\}$ $\quad G_4 = \{e, (13)\}$

- S_4 acting on itself by conjugation.

every $g \in S_4$ is conjugate to one of these 5 examples.

Orbits

e

(12)

(123)

$(12)(34)$

(1234)

Orbit of $e = \{geg^{-1} \mid g \in S_4\} = \{e\}$

Orbit of $(12) = \{g(12)g^{-1} \mid g \in S_4\} = \{(ab)\}$ 6 elts.

Orbit of $(123) = \{3\text{-cycles in } S_4\}$ 8 elts.

$(12)(34)$, $(13)(24)$, $(14)(23)$ 3 elts

(1234) 6

Stabilizer of (12)

$G_{(12)} = \{g \mid g(12)g^{-1} = (12)\}$

$\{e, (12), (34), (12)(34)\}$ ☆

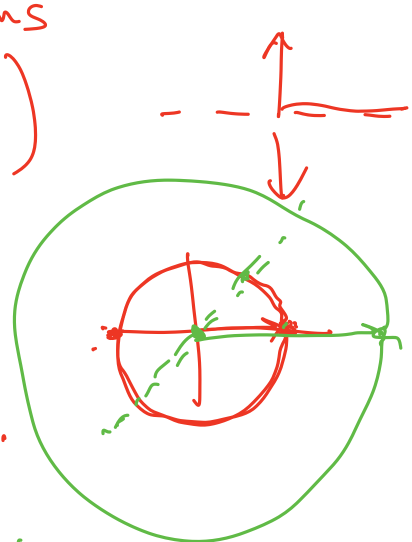
$G_{(123)} = \{g \mid g(123)g^{-1} = (123)\}$

$\{e, (123), (132)\}$. ☆

- The orthogonal group $O(2)$ acting on the plane.

$SO_2 \subseteq O_2$ rotations

reflections $\to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Orbit of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$O(2)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = unit circle.

$O(2)\begin{bmatrix} n \\ 0 \end{bmatrix}$ = circle of radius n.

$O(2)\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

Stabilizer of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\left\{ e, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = G_{\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}}$

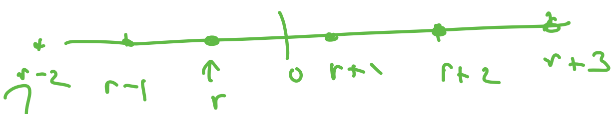
- The subgroup $\mathbb{Z}$ acting on $\mathbb{R}$.

$\mathbb{Z}$ on $\mathbb{R}$.

Orbit of $\{0\}$.

$\{0+n \mid n \in \mathbb{Z}\} = \mathbb{Z} \subseteq \mathbb{R}$.

orbit of $\{\pi\}$



$\{n+\pi \mid n \in \mathbb{Z}\}$

$\mathrm{Stab}(x) = \{n \mid x+n = x\} = \{0\}$

- The permutation group S_n acting on strings of 0's and 1's of length n by permuting their positions.

$$X = \{\overbrace{00100}^{n \text{ entries}}, 100\cdots1\} = \overbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}^{n \text{ times}}$$

S_n acts

$n=3$ $\quad S_3$

(12)

1	2	3
0	0	0
⋮		
1	1	1

$000 \xrightarrow{(12)} 000$

$010 \xleftrightarrow{(12)} 100$

$011 \xleftrightarrow{(12)} 101$

$010 \xleftrightarrow{(123)} (001)$

Orbit of $\quad (01\cdots1001\cdots01) = x$

$(0\cdots0\cdots01\cdots1)$

$x \underset{G \text{ equivalent}}{\sim}$

$x \sim y \iff$ they have same # of 1's.

n orbits

0	1's	$00\cdots0$
1	1	$00\cdots01$
2	1's	$00\cdots11$
⋮		

$n=5$

$x = \underbrace{000}\underbrace{11}$ $\qquad G_x = \underset{1,2,3}{S_3} \times \underset{4,5}{S_2}$

Stab of $\quad \underbrace{\overbrace{0\cdots0}^{k}}\ \underbrace{\overbrace{1\cdots1}^{n-k}}$

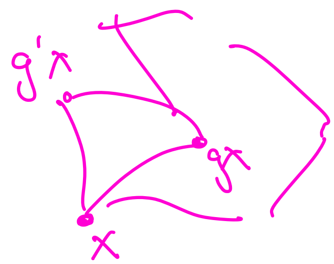
stab is $\quad S_k \times S_{n-k}$

Proposition: Let $x \in X$. The map

$$p_x : G \to X$$

defined by $p(g) = gx$ gives a bijection between the cosets of the stabilizer subgroup G_x and the orbit Gx. In particular $[G : G_x]$ and $|Gx|$ are either both infinite or both finite, and if both finite then $|Gx| = [G : G_x]$.

Proof: Cosets of stabilizer of x $\xLeftrightarrow{\text{bijective}}$ elements of orbit of x.



$x \in X$.

$p_x : G \to X$

$p_x(g) = gx$.

By definition image = orbit of x = Gx.

$g \in G_x \quad p_x(g) = gx = x$.

$\tilde{p}$: left cosets of G_x in G $\longrightarrow$ Orbit of x in X.

$$\boxed{\tilde{p}(gG_x) = gx.}$$

$g_1 \in gG_x \quad g_1 = gh \quad h \in G_x$

$g_1 x = g\underbrace{hx} = gx$

If $x' \in Gx$ then $x' = gx$ for some g.

$\tilde{p}(gG_x) = gx = x'$. SURJECTIVE

If $\tilde{p}(g_1 G_x) = \tilde{p}(g_2 G_x)$,

then $g_1 x = g_2 x$

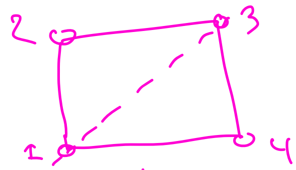
so $g_2^{-1} g_1 x = x$

$g_2^{-1} g_1 \in G_x$

$g_1 \in g_2 G_x$ $\qquad$ $g_1 G_x = g_2 G_x$

injectivity.

D_4



just one orbit $\{1,2,3,4\}$

$\underset{8}{D_4} \cdot 1 = \underbrace{\{1,2,3,4\}}_{4}$.

Stab of $1 = \{e, (24)\}$.