

Solvable Groups

Definition: A group is called *solvable* if there is a sequence of subgroups

$$G \supset H_1 \supset H_2 \supset \dots \supset \{0\}$$

$H_2 \trianglelefteq H_1$ normal
 $H_3 \trianglelefteq H_2$ normal
 $H_2/H_1, H_3/H_2$ abelian

where each subgroup H_{i+1} is normal in H_i and if the quotients H_i/H_{i+1} are abelian.

Examples

- Abelian groups are solvable

G abelian
 $G = \{0\}$

- S_3 and S_4 are solvable.

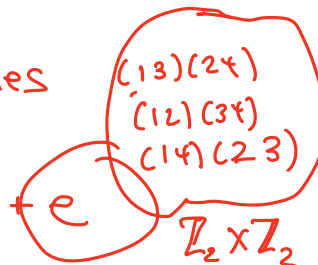
S_4 24 els

$A_4 \trianglelefteq S_4$

A_4 : 3-cycles
8

$A_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$A_4 / (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_3$



- The dihedral groups are solvable.

group of rotations $\trianglelefteq D_n \cong \mathbb{Z}_n$

$D_n / \langle R \rangle \cong \mathbb{Z}_2$

$\langle \sigma \rangle \cong \mathbb{Z}_3 \trianglelefteq \mathbb{Z}_2 \times \mathbb{Z}_2 \trianglelefteq A_4 \trianglelefteq S_4$
 cycle of order 3

- S_5 is not solvable.

$S_5 \cong A_5$
 120 60

A_5 has nontrivial normal subgroups
 A_5 is simple.