Solvable Groups

Definition: A group is called *solvable* if there is a sequence of subgroups

$$G \supset H_1 \supset H_2 \supset \cdots \supset \{0\}$$

$H_2 \subseteq H_1$ normal
$H_3 \subseteq H_2$ normal
H_2/H_1, H_3/H_2 . abelian

where each subgroup H_{i+1} is normal in H_i and if the quotients H_i/H_{i+1} are abelian.

Examples

- Abelian groups are solvable

  G abelian

  $G \supseteq \{0\}$

- S_3 and S_4 are solvable.

  S_4 24 elts

  $A_4 \subseteq S_4$

  A_4: 3-cycles 8

  $(13)(24)$, $(12)(34)$, $(14)(23)$, e

  $\mathbb{Z}_2 \times \mathbb{Z}_2$

  $A_4 \supseteq \mathbb{Z}_2 \times \mathbb{Z}_2$

  $A_4/(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq \mathbb{Z}_3$

  $\{0\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq A_4 \subseteq S_4$

  cycle of order 3

- The dihedral groups are solvable.

  group of rotations $\le D_n$

  $D_n/\langle R \rangle \simeq \mathbb{Z}_2$

- S_5 is not solvable.

  $S_5 \supseteq A_5$

  120 60

  A_5 has nonnontrivial normal subgps

  A_5 is simple.