Wallpaper and Crystals

Lattices

Definition: A lattice L in $\mathbb{R}^n$ is a subgroup consisting of all integer linear combinations of a basis of $\mathbb{R}^n$. That is,

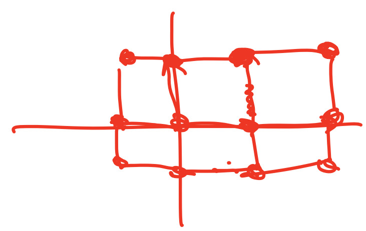
$$L = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n : n_i \in \mathbb{Z}\}$$

where $x_1, \ldots, x_n$ are a linearly independent set of vectors in $\mathbb{R}^n$

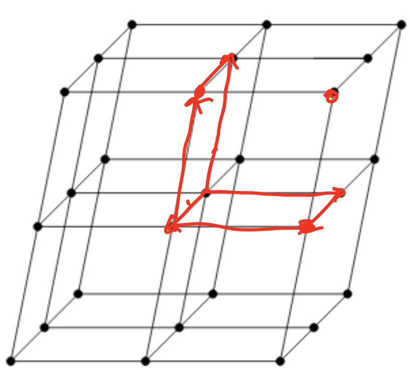
Examples

- $n = 2$, $x_1 = \mathbf{i}$, $x_2 = \mathbf{j}$. $\mathbb{R}^2$ $\{\hat{i}, \hat{j}\}$

  $L = \{a\hat{i} + b\hat{j} \mid a, b \in \mathbb{Z}\}$



- $n = 3$ (See Figure 12.17 in the text):



Proposition: A lattice in $\mathbb{R}^n$ is an abelian group that is isomorphic to $\mathbb{Z}^n$.

$\mathbb{Z}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{Z}\}$

$f: \mathbb{Z}^n \longrightarrow L \leq \mathbb{R}^n$ $\qquad L = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in \mathbb{Z}\}$

$\{x_i\}$ a basis

$f((a_1, \ldots, a_n)) = \sum_{i=1}^{n} a_i x_i$

$f((a_1, \ldots, a_n)) + f((b_1, \ldots, b_n)) = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i$

$= \sum_{i=1}^{n} (a_i + b_i) x_i = f((a_1+b_1, \ldots, a_n+b_n))$

Suppose $f((a_1, \ldots, a_n)) = 0$.

then $\sum_{i=1}^{n} a_i x_i = 0$. But $\{x_i\}$ are a basis so this means all $a_i = 0$.

so $(a_1, \ldots, a_n) = (0, \ldots, 0)$

$l \in L$ $\quad l = \sum a_i x_i$ then $f(a_1, \ldots, a_n) = l$.

$\sum a_i x_i$.

Definition: Let L be a lattice in $\mathbb{R}^n$. The *automorphism group* of L is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices g such that $gL = L$. This group is called the *unimodular group.*

Proposition: The unimodular group is the group $\mathrm{GL}_n(\mathbb{Z})$ consisting of $n \times n$ matrices with integer entries and determinant ± 1.

$\mathbb{R}^2$ $L = \{a\hat{i} + b\hat{j}\}$
$= \{a\hat{i} + b(-\hat{j})\}$
$= \{a\hat{i} + b(\hat{i}+\hat{j})\}$

$GL_n(\mathbb{Z}) = \{g \mid gL = L\}$.

$n=2$ case.

L spanned by e_1, e_2.

$L = \{ae_1 + be_2\}$,

$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ $gL \stackrel{?}{=} L$.

$g(ae_1+be_2) = ag(e_1) + bg(e_2)$

$L \subseteq gL$? $gL = \{ag(e_1) + bg(e_2) \mid a, b \in \mathbb{Z}\}$

$L \subseteq gL \iff e_1$ and $e_2 \in gL$.

$e_1 = ag(e_1) + bg(e_2)$
$e_2 = cg(e_1) + dg(e_2)$ for $a,b,c,d \in \mathbb{Z}$.

$gL \subseteq L$?

$g(e_1) = \frac{1}{ad-bc}(de_1 - be_2)$

$g(e_2) = \frac{1}{ad-bc}(-ce_1 + ae_2)$

$gL = L$
$\Updownarrow$
u has $\mathbb{Z}$-coeffs
u^{-1} has $\mathbb{Z}$-coeffs

$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$u^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\Delta = ad-bc$.

$\det(uu^{-1}) = 1$
$\det(u)\det(u^{-1}) = 1$

$ad-bc = \pm 1$
$g \in GL_2(\mathbb{Z})$
$\det(g) = \pm 1$

Crystallography

Crystals are objects whose underlying atoms or molecules are organized in a lattice structure. Two such crystals are equivalent if there is a Euclidean symmetry $g \in E(3)$ that transforms one into another.

One can classify crystals by giving the subgroup of $E(3)$ consisting of Euclidean symmetries that preserve it. This is called the *symmetry group* of the crystal. One can show that there are 230 possible such subgroups of $E(3)$ and thus 230 different classes of crystals.

To get a feel for this we will look at "crystals" in 2-dimensions.

An example from Escher

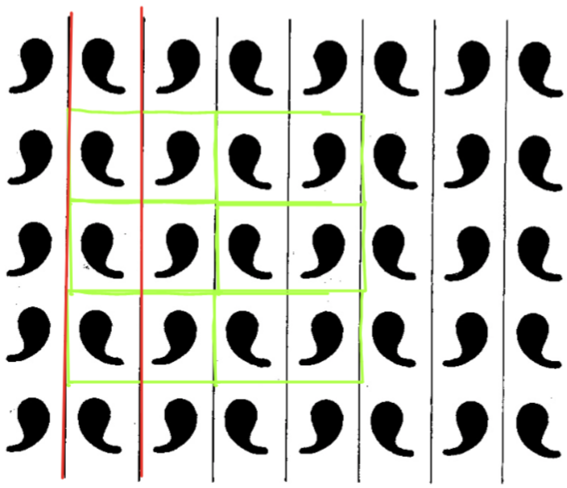
This image is taken from the book *Fantasy and Symmetry*, by Caroline MacGillavry, Abrams Publishers, NYC, 1976.

Figure 1: Escher

Another example

This image is taken from the website Plane Symmetry at

`www.york.ac.uk/depts/maths/histstat/symmetry/welcome.htm`



pm

Wallpaper groups

Remember that elements of $E(2)$ are pairs (A, a) where A is an orthogonal matrix (hence a rotation or a reflection) and a is a vector in $\mathbb{R}^2$. The pairs $(1, a)$ form the translation subgroup T isomorphic to $\mathbb{R}^2$. There is a surjective homomorphism

$$(A,a)\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + a$$

$$\text{translations} \to E(2) \xrightarrow{\pi} O(2)$$

that sends (A, a) to A.

$$(A,a)(B,b) = (AB, Ab+a)$$

Definition: If H is a subgroup of $E(2)$, the translation subgroup of H is $H \cap T$ and the space group of H is the image of H in $O(2)$ under this homomorphism.

$$H \subseteq E(2)$$

$$H \cap T \qquad \pi(H) \subseteq O(2)$$

Definition: A subgroup H of $E(2)$ is called a wallpaper group or a plane group if the translations $H \cap T$ in H form a lattice in $\mathbb{R}^2$ and the space group of H is finite.

Another look at Escher

Figure 2: Escher

$a \to$ shift 1 right

$b \to$ shift 1 up.

$\underbrace{a^n b^m}$ $n, m \in \mathbb{Z}$

R $ab = ba$

$R^2 = e$

$$R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$a \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix}$$

$$a^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$b \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y+1 \end{bmatrix}$$

$$b^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y-1 \end{bmatrix}$$

$$Ra^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = R \begin{bmatrix} x-1 \\ y \end{bmatrix} = \begin{bmatrix} 1-x \\ -y \end{bmatrix} \quad \|$$

$$aR \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} 1-x \\ -y \end{bmatrix}$$

$$\left. \begin{array}{l} Ra^{-1} = aR \\ Rb^{-1} = bR \end{array} \right\}$$

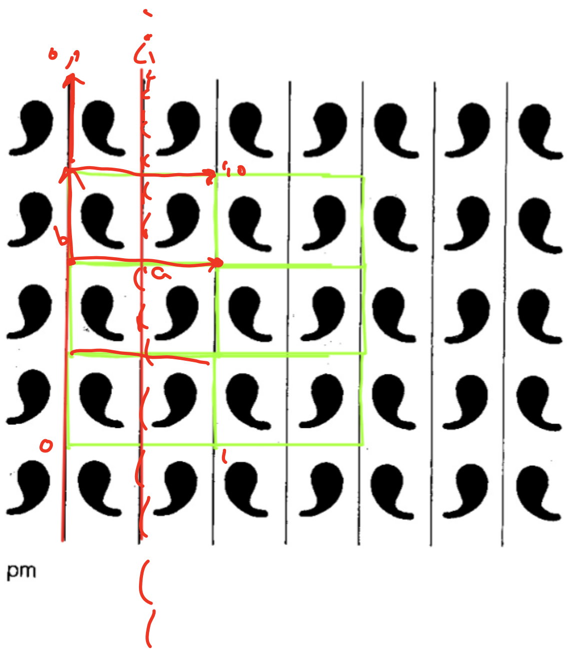
$$H = \left\{ \begin{array}{l} a^n b^m \\ R a^n b^m \end{array} \right.$$

point group $\mathbb{Z}/2$

$\langle R \rangle$

$R^2 = e$

Another look at the commas



pm

a shift left/right

b shift up/down

m

$$a\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+1\\y\end{bmatrix} \qquad ab=ba$$

$$b\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y+1\end{bmatrix}$$

$$m\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1-x\\y\end{bmatrix}$$

$$mb=bm$$

$$mb\begin{bmatrix}x\\y\end{bmatrix}=m\begin{bmatrix}x\\y+1\end{bmatrix}=\begin{bmatrix}1-x\\y+1\end{bmatrix} \quad \|$$

$$bm\begin{bmatrix}x\\y\end{bmatrix}=b\begin{bmatrix}1-x\\y\end{bmatrix}=\begin{bmatrix}1-x\\y+1\end{bmatrix}$$

$$ma^{-1}\begin{bmatrix}x\\y\end{bmatrix}=m\begin{bmatrix}x-1\\y\end{bmatrix}=\begin{bmatrix}1-(x-1)\\y\end{bmatrix}=\begin{bmatrix}2-x\\y\end{bmatrix}$$

$$am\begin{bmatrix}x\\y\end{bmatrix}=a\begin{bmatrix}1-x\\y\end{bmatrix}=\begin{bmatrix}2-x\\y\end{bmatrix} \quad \|$$

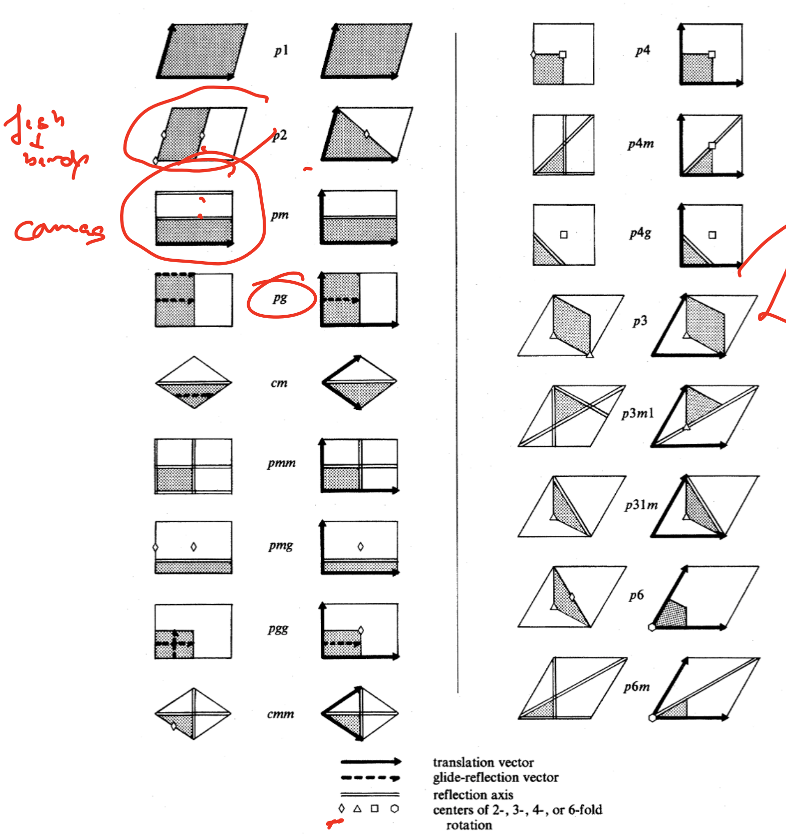
$$ma^{-1}=am$$

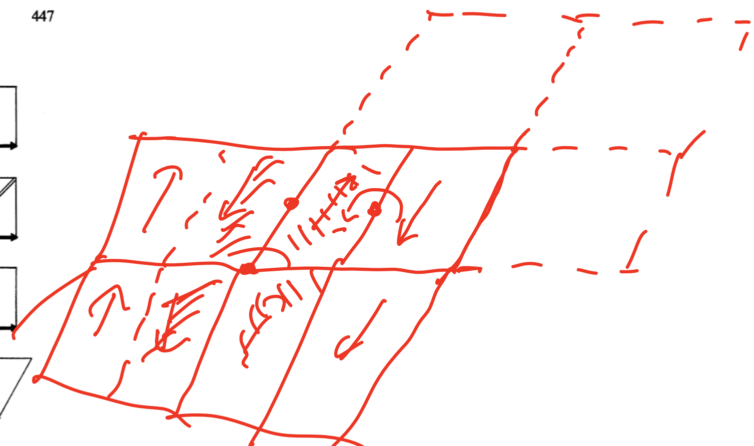
$$\begin{cases}a^n b^m\\ m a^n b^m\end{cases} \qquad m^2=e \qquad \text{pt gp}=\mathbb{Z}_2 \qquad T=\langle a,b\rangle.$$

The full list

This is Chart 5, from "The Plane Symmetry Groups: Their Recognition and Notation" by Doris Schattschneider, American Mathematical Monthly, Jun-Jul 1978, Vol. 85, No. 6, pp 439-450. This article also contains pictures illustrating all 17 patterns.

GENERATORS FOR THE PLANE SYMMETRY GROUPS





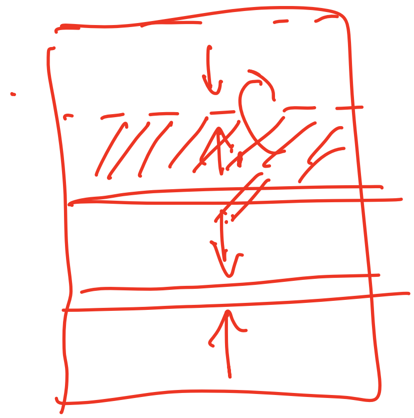
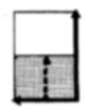
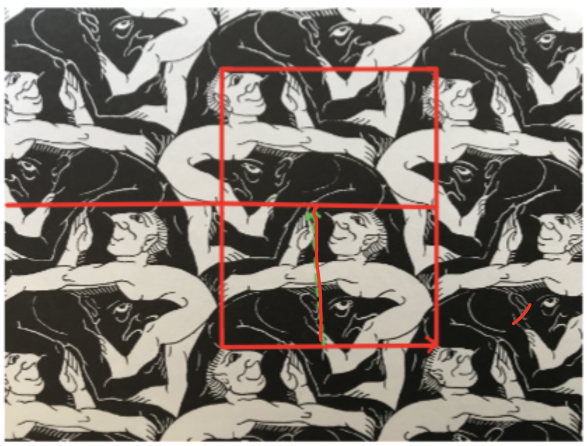


CHART 5. For each group, two sets of generators are indicated relative to a lattice unit containing a shaded generating region. A minimal set of generators is shown at the left, while a set of generators which includes the lattice unit translation vectors is shown at the right.

Another Escher (pg symmetry)





pg

For a proof that there are exactly 17 possibilities see *The 17 plane symmetry groups*, by RLE Schwarzenberger, The Mathematical Gazette, Volume 58, No. 404, June 1974, pp. 123-131.

But neither the passive contemplation of wallpaper patterns, nor the passive contemplation of abstract definitions, is mathematics: the latter is above all an activity in which definitions are used to obtain concrete results.

A partial result

Proposition: Suppose the plane group has no reflections or glide reflections. Then there are only 5 possibilities classified by whether or not the point group is $\mathbb{Z}_n$ for $n = 1, 2, 3, 4, 6$.

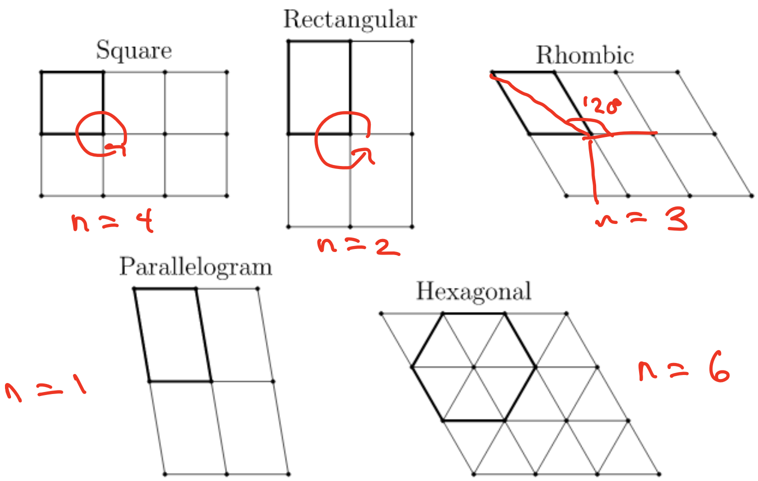


Figure 12.21. Types of lattices in $\mathbb{R}^2$

Lemma: A lattice contains a shortest vector.

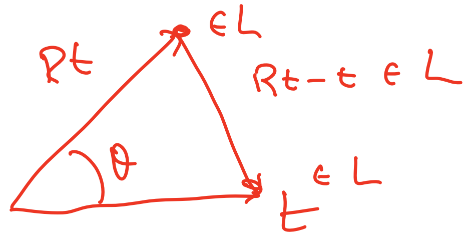
L a lattice

$RL \subseteq L$

Know gp of rotations is cyclic generated by a rotation through $2\pi/n$ radians for $n = 1, 2, \ldots$

R be rotation through $2\pi/n$.

t shortest vector.



$\|Rt\|^2 = \|t\|^2$

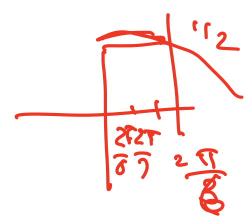
$$\|Rt - t\|^2 \geq \|t\|^2$$

$$\|Rt\|^2 - 2\langle Rt, t\rangle + \|t\|^2 \geq \|t\|^2$$

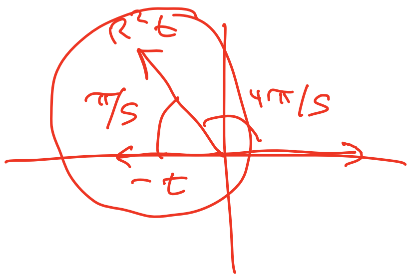
$$\|t\|^2 = \|t\|^2 + \|t\|^2 - \|t\|^2 \geq 2\langle Rt, t\rangle = 2\|Rt\|\|t\|\cos\theta$$
$$= 2\|t\|^2\cos\theta$$

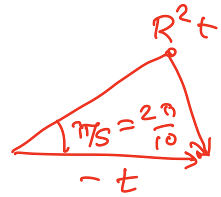
$$\cos\theta \leq \frac{1}{2}$$

$$\cos\frac{2\pi}{n} \leq \frac{1}{2} \qquad \underline{n = 1, 2, 3, 4, 5, 6}$$



$\cos\frac{2\pi}{5} \leq \frac{1}{2}$ so that's not a contradiction..





$$\underbrace{R^2 t - (-t)}_{\omega} \in L$$

$$\cos\frac{2\pi}{10} > \frac{1}{2}$$

$$\|w\| < \|t\|$$

contradiction to shortest vector.