Quick review of linear algebra

Matrices yield linear maps

$f(x+y) = f(x) + f(y)$

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if $f(ax + by) = af(x) + bf(y)$ for all $x, y \in \mathbb{R}^n$ and all $a, b \in \mathbb{R}$.

$f(ax) = a f(x)$

An $m \times n$ matrix A yields a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ via matrix multiplication $x \mapsto Ax$.

A $m \times n$ matrix

$x \in \mathbb{R}^n$ $\quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \underset{m\times n \;\; n \times 1}{Ax} \longrightarrow \underset{m \times 1}{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}} \in \mathbb{R}^m$

Examples

$A(x+y) = Ax + Ay$
$A(cx) = cAx.$

- The identity matrix/identity linear map from $\mathbb{R}^n$ to itself.

  $f : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad \underset{n \times n}{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}$

- The zero map from $\mathbb{R}^n \to \mathbb{R}^m$

  $f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad f(x) = 0$ for all x

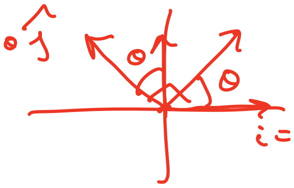
  $m \times n \begin{pmatrix} 0 & \cdots & 0 \\ & \vdots & \\ 0 & 0 & 0 \end{pmatrix}$

- The rotation matrix $M(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

  $\theta \in \mathbb{R}$

  $M(\theta) \overset{\hat{i}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\qquad M(\theta) \overset{\hat{j}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

  $\cos\theta\, \hat{i} + \sin\theta\, \hat{j}$



$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Every linear map comes from a matrix

Given a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, we can associate to it an $m \times n$ matrix A with entries (a_{ij}) by computing

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{f}_i$$

where $\mathbf{e}_j$ and $\mathbf{f}_i$ are the n- and m- dimensional column vectors with a one in position j (resp. i) and zeros elsewhere.

$$\mathbb{R}^n \ni e_j = \begin{array}{c} \\ \\ j \to \\ \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbb{R}^m \ni f_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad T(e_j) = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 f_1 + x_2 f_2 + \dots + x_m f_m$$

$$T(e_j) = \sum_{i=1}^{m} a_{ij} f_i \quad \text{for } j = 1, \dots, n.$$

$$A = (a_{ij})$$

$$\begin{pmatrix} & a_{1j} & \\ & a_{2j} & \\ & \vdots & \\ & a_{mj} & \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} = T(e_j)$$

$$A(e_1) = 2f_1 + f_2$$
$$A(e_2) = 5f_1 - 6f_2$$

$$A = \begin{pmatrix} 2 & 5 \\ 1 & -6 \end{pmatrix} \qquad Ae_1 = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2f_1 + f_2$$

Matrix Multiplication is composition of linear maps

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $V : \mathbb{R}^m \to \mathbb{R}^p$ are linear maps with associated matrices A and B, then the matrix associated to the composition VT $\cancel{TV}$ is the matrix product $\cancel{AB}$. BA

$$V \circ T : \mathbb{R}^n \longrightarrow \mathbb{R}^p$$

$$A \Longleftrightarrow T \quad m \times n$$

$$B \Longleftrightarrow V \quad p \times m$$

$$BA \qquad V \circ T \quad p \times n$$

$$A \Longleftrightarrow T$$

$$T(\underline{e_j}) = \sum_{i=1}^{m} a_{ij} \underline{f_i} \qquad j = 1, \dots, n$$

$$V(f_i) = \sum_{k=1}^{p} b_{ki} t_k \qquad t_k \quad k = 1, \dots, p$$

$$VT(\underline{e_j}) = V\left(\sum_{i}^{m} a_{ij} f_i\right) = \sum_{i}^{m} a_{ij} V(f_i)$$

$$= \sum_{i}^{m} a_{ij} \sum_{k=1}^{p} b_{ki} t_k$$

$$= \sum_{k=1}^{p} \Big(\underbrace{\sum_{i=1}^{m} b_{ki} a_{ij}}_{c_{kj}} \Big) t_k$$

$$c_{kj} = \sum_{i=1}^{m} b_{ki} a_{ij}$$

$$C = BA.$$

A linear map is bijective if its matrix is invertible

- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is bijective then its inverse is also linear and the associated matrix is the inverse matrix A^{-1}. Conversely if the associated matrix is invertible then T is bijective. In particular the inverse of a bijective linear map is ~~bijective~~. linear.

all correspond to the same thing.

$$\left\{\begin{array}{l}\text{invertible linear maps}\\ \text{invertible matrices}\\ \text{matrices } A \text{ with } \det(A) \neq 0\end{array}\right.$$

A matrix is invertible if and only if it has nonzero determinant

The inner product (dot product)

Definition: The Euclidean inner product on $\mathbb{R}^n$ is the dot product

$$(\sum_{i=1}^{n} a_i \mathbf{e}_i) \cdot (\sum_{i=1}^{n} b_i \mathbf{e}_i) = \sum_{i=1}^{n} a_i b_i.$$

$(3,5)\cdot(2,7)$

$= 3\cdot 2 + 5\cdot 7$

$= 41$

If $x, y \in \mathbb{R}^n$ this is also written $\langle x, y \rangle$.

$\langle x, y \rangle = x \cdot y$

Properties of the inner product

Proposition: The inner product is:

- symmetric, so $\langle x, y\rangle = \langle y, x\rangle$

$$\left(\sum a_i e_i\right) \cdot \left(\sum b_i e_i\right) = \sum a_i b_i$$

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

- bilinear, so $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ and $\langle ax, y\rangle = a\langle x, y\rangle$ for $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

$\Leftarrow$ $x = \sum a_i e_i$, $y = \sum b_i e_i$, $z = \sum c_i e_i$

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

$$\sum a_i (b_i + c_i) = \sum a_i b_i + \sum a_i c_i$$

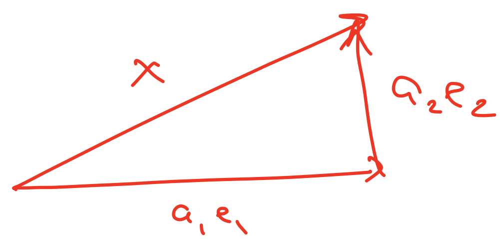
$$ax \cdot y = a \sum a_i b_i$$

- positive definite, so $\langle x, x\rangle \geq 0$ for all x and is zero only if $x = 0$.

$$\langle x, x\rangle = \sum a_i^2 \geq 0$$

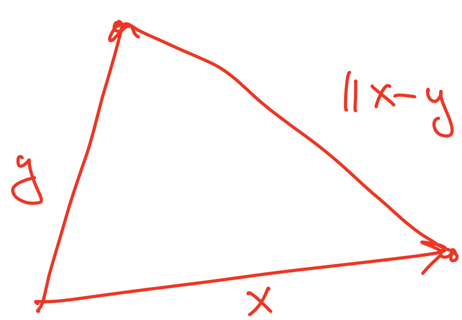
$x = \sum a_i e_i$ $\qquad \sum a_i^2 = 0 \iff \text{all } a_i = 0 \iff x = 0$

Given a vector x, the quantity $x \cdot x = \|x\|^2$ is called the norm of x; geometrically it is the length of the vector x.



$\|x\|^2 = a_1^2 + a_2^2$

Given two vectors x and y, the quantity $(x - y) \cdot (x - y) = \|x - y\|^2$ is the square of the Euclidean distance between x and y.



$\|x-y\|^2$ = squared distance from x to y