Isomorphism Theorems

The canonical homomorphism

Let G be a ~~sub~~group and H be a normal subgroup of G. Then the map

$$\phi : G \to G/H$$

elts are cosets gH

$(g_1H)(g_2H) = g_1g_2H$

defined by $\phi(g) = gH$ is a homomorphism called the *natural homomorphism* or the *canonical homomorphism.*

$\varphi(g) = gH.$

$\varphi(g_1g_2) = g_1g_2H$

$= (g_1H)(g_2H)$

$= \varphi(g_1)\varphi(g_2)$

$K(\varphi) = \{ g \mid \varphi(g) = H \}$

$= \{ g \mid gH = H \}$

$= \{ g \in H \}$

$\ker(\varphi) = H.$

Examples

- $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ $\quad n \geq 1$

  $a \longmapsto a + n\mathbb{Z}$

- $S_n \to S_n/A_n = \mathbb{Z}_2$

  $\varphi(\sigma) = \sigma A_n$

  $A_n = \{\text{even perm}\}$

  $\tau A_n = \{\text{odd perms}\}$

  τ is any odd perm.

- $\mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{R}) = \mathbb{R}^\times$ (det)

  $g\,\mathrm{SL}_2(\mathbb{R}) = \{ h \mid \det(h) = \det(g) \}.$

The first isomorphism theorem

Theorem: (First isomorphism theorem) Suppose

- $\psi : G \to H$ is a group homomorphism
- K is the kernel of ψ
- $\phi : G \to G/K$ is the canonical homomorphism.

Then there exists a unique isomorphism $\eta : G/K \to \psi(G) \subset H$ such that $\psi = \eta\phi$.

$$
\begin{tikzcd}
K \leq G \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & \psi(G) \leq H \\
& G/K \arrow[ur, "\eta"'] &
\end{tikzcd}
$$

Figure 1: First isomorphism theorem diagram

The first isomorphism theorem (in a slightly different context) is due to Emmy Noether in her paper *Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionkoerpern* (Abstract Foundations of Ideal Theory in Number Fields and Function Fields), Math. Annalen, 1927.

In words: every homomorphism ψ from G to H is a composition of two steps:

- first, you take the natural map from G to the quotient G/K where K is the kernel of ψ;
- then you have an isomorphism from G/K to a subgroup of H.

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\varphi"'] & H \\ G/K \arrow[r, phantom, "\cong"] & \psi(H) \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

$K = \ker(\psi)$.

Or, put another way, the image of a homorphism of a group G is isomorphic to a quotient of G.

Another look at the sign map $S_n \to \mathbb{Z}_2$.

$\psi(\sigma) = 0 \quad \sigma$ even

$\psi(\sigma) = 1 \quad \sigma$ odd

$K = \ker(\varphi) = A_n$.

$$\begin{tikzcd}
S_n \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & \mathbb{Z}_2 \arrow[d, leftrightsquigarrow] \\
 & \underline{S_n / A_n}
\end{tikzcd}
\qquad \operatorname{im}(\psi) = \mathbb{Z}_2$$

Another look at the determinant map from $\mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$.

$$\begin{tikzcd} K \subseteq \mathrm{Gl}_2(\mathbb{R}) \arrow[r, "\psi = \det"] \arrow[dr] & \underline{\mathbb{R}^*} \arrow[d, equal, "\wr"] \\ & \underline{\mathrm{GL}_2(\mathbb{R}) / \mathrm{SL}_2(\mathbb{R})} \end{tikzcd}$$

$\psi(a) = \mathbb{R}^*$

$K = \mathrm{SL}_2(\mathbb{R})$

Let G be a group and let $g \in G$ be an element. Consider the homomorphism

$$\psi\cancel{\phi} : \mathbb{Z} \to G$$

given by $\overset{\psi}{\cancel{\phi}}(g) = g^n$.

$g \in G$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\psi(g)=g^n"] & G \end{tikzcd}$$

$\text{image}(\psi) = \{g^n \mid n \in \mathbb{Z}\} = \langle g \rangle.$

$\text{Ker}(\psi) = \{n \mid g^n = e\}$

$g^n = e \iff \text{order}(g) \mid n$ (or g has infinite order).

Suppose g has finite order.

$\text{Ker}(\psi) = \text{order}(g)\mathbb{Z}.$

$$\begin{tikzcd} \text{order}(g)\mathbb{Z} \subseteq \mathbb{Z} \arrow[r, "\psi"] \arrow[dr] & \langle g \rangle \subseteq G \\ & \mathbb{Z}/\text{order}(g)\mathbb{Z} \arrow[u, "\cong"'] \end{tikzcd}$$

$\langle g \rangle \cong \mathbb{Z}/\text{order}(g)\mathbb{Z}$

$g^n \leftarrow n$

Another look at the map $\psi: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ when $m|n$.

$$\mathbb{Z}/12\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$$

$$a+12\mathbb{Z} \longrightarrow a+3\mathbb{Z}$$

$$3\mathbb{Z}/12\mathbb{Z} \subseteq \mathbb{Z}/12\mathbb{Z}$$

$$3\mathbb{Z}/12\mathbb{Z} = \ker(\psi)$$

$$\begin{tikzcd}
\mathbb{Z}/12\mathbb{Z} \arrow[r, "\psi:"] \arrow[dr] & \mathbb{Z}/3\mathbb{Z} \arrow[d, phantom, "\wr"] \\
& (\mathbb{Z}/12\mathbb{Z})/(3\mathbb{Z}/12\mathbb{Z})
\end{tikzcd}$$

Theorem: (First isomorphism theorem) Suppose

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$$\begin{tikzcd} G \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & H \\ & G/K \arrow[ur, "\eta"'] & \end{tikzcd}$$

Figure 1: First isomorphism theorem diagram

Want: $\psi(g) = \eta\phi(g)$

Def: $\eta(gK) = \psi(g) \in H$.

Check that this η works.

(a) $\eta(gK) = \eta(g'K)$ if $gK = g'K$?

$gK = g'K$ ~~then~~ $\Leftrightarrow$ $g = g'k$ for $k \in K$.

$\eta(gK) = \psi(g) = \psi(g'k) = \psi(g')\psi(k) = \psi(g')$ since $k \in \mathrm{Ker}(\psi)$ $\psi(k) = e$.

$\eta(g'K) = \psi(g')$

(b) $\eta((g_1K)(g_2K)) = \eta(g_1g_2K) = \psi(g_1g_2)$

$\eta(g_1K)\eta(g_2K) = \psi(g_1)\psi(g_2)$ since ψ a homomorphism

(c) injective. $\eta(g_1K) = \eta(g_2K)$

means $\psi(g_1) = \psi(g_2)$

$\psi(g_1)\psi(g_2)^{-1} = e$

$\psi(g_1 g_2^{-1}) = e$

$g_1 g_2^{-1} \in \ker(\psi) = K$

$g_1 K = g_2 K$.

d. If $h \in \psi(G)$, $h = \psi(g)$ for some $g \in G$.

$\eta(gK) = \psi(g) = h$.