

Isomorphism Theorems

The canonical homomorphism

Let G be a ~~sub~~group and H be a normal subgroup of G . Then the map

$$\phi : G \rightarrow \underbrace{G/H}_{\text{elts are cosets } gH}$$

$$(g_1 H)(g_2 H) = g_1 g_2 H \leftarrow$$

defined by $\phi(g) = gH$ is a homomorphism called the *natural homomorphism* or the *canonical homomorphism*.

Examples

- $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n \quad n \geq 1$
 $a \mapsto a+n\mathbb{Z}$

- $S_n \rightarrow S_n/A_n = \mathbb{Z}_2$
 $\phi(\sigma) = \sigma A_n$

$A_n = \{\text{even perms}\}$
 $\tau A_n = \{\text{odd perms}\}$
 τ is any odd perm.

- $GL_2(\mathbb{R}) \xrightarrow{\det} GL_2(\mathbb{R})/SL_2(\mathbb{R}) = \mathbb{R}^\times$

$$g SL_2(\mathbb{R}) = \{ h \mid \det(h) = \det(g) \}$$

$\phi(g) = gH$
 $\phi(g_1 g_2) = g_1 g_2 H = (g_1 H)(g_2 H) = \phi(g_1) \phi(g_2)$
 $\text{Ker}(\phi) = \{ g \mid \phi(g) = H \} = \{ g \mid gH = H \} = \{ g \in H \}$
 $\text{Ker}(\phi) = H$

The first isomorphism theorem

Theorem: (First isomorphism theorem) Suppose

- $\psi : G \rightarrow H$ is a group homomorphism
- K is the kernel of ψ
- $\phi : G \rightarrow G/K$ is the canonical homomorphism.

Then there exists a unique isomorphism $\eta : G/K \rightarrow \underline{\psi(G)} \subset \underline{H}$ such that $\psi = \eta\phi$.

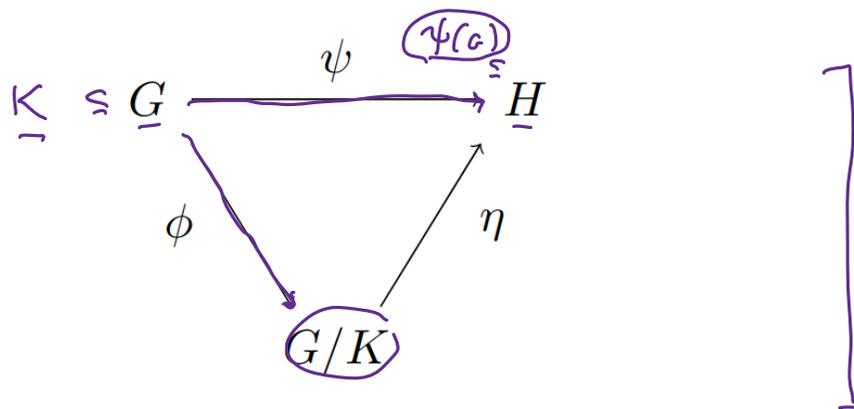
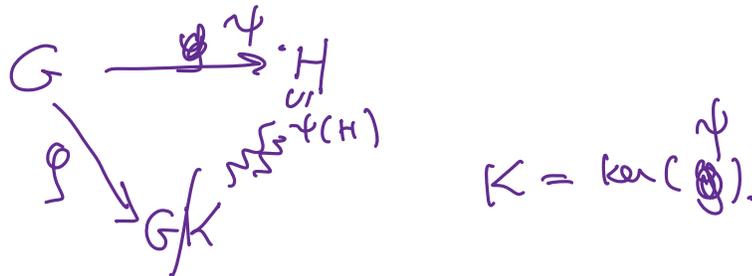


Figure 1: First isomorphism theorem diagram

The first isomorphism theorem (in a slightly different context) is due to Emmy Noether in her paper *Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionkoerpern* (Abstract Foundations of Ideal Theory in Number Fields and Function Fields), Math. Annalen, 1927.

In words: every homomorphism ψ from G to H is a composition of two steps:

- first, you take the natural map from G to the quotient G/K where K is the kernel of ψ ;
- then you have an isomorphism from G/K to a subgroup of H .



Or, put another way, the image of a homomorphism of a group G is isomorphic to a quotient of G .

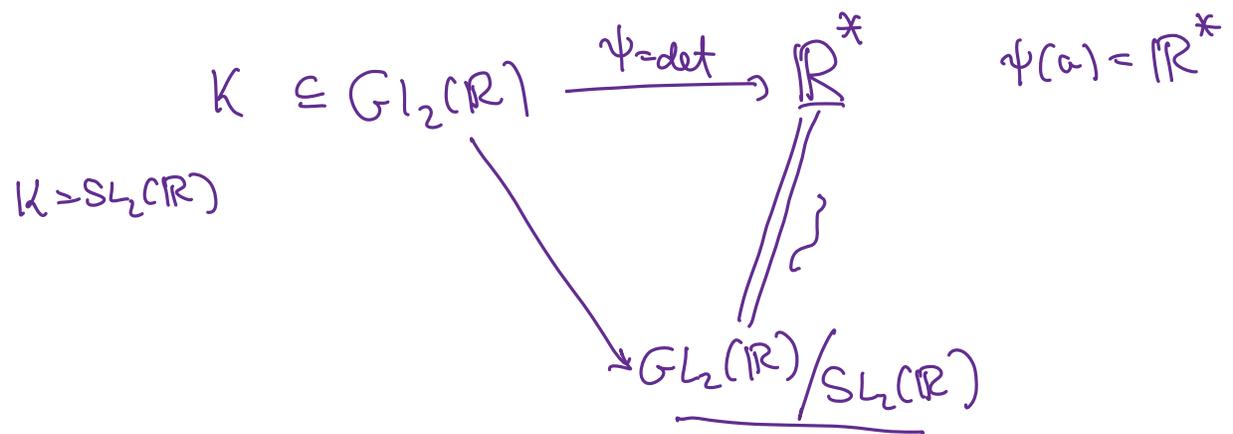
Another look at the sign map $S_n \rightarrow \mathbb{Z}_2$.

$$\begin{aligned} \psi(\sigma) &= 0 && \sigma \text{ even} \\ \psi(\sigma) &= 1 && \sigma \text{ odd} \end{aligned}$$

$$K = \ker(\psi) = A_n.$$

$$\begin{array}{ccc} S_n & \xrightarrow{\psi} & \mathbb{Z}_2 \\ \downarrow \varphi & & \downarrow \cong \\ \underline{S_n / A_n} & & \mathbb{Z}_2 \end{array} \quad \text{im}(\psi) = \mathbb{Z}_2$$

Another look at the determinant map from $GL_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$.



Let G be a group and let $g \in G$ be an element. Consider the homomorphism

$$\psi: \mathbb{Z} \rightarrow G$$

given by $\psi(g) = g^n$.

$$\mathbb{Z} \xrightarrow{\psi(g) = g^n} G$$

$$g \in G$$

$$\begin{aligned} \text{image}(\psi) &= \{g^n \mid n \in \mathbb{Z}\} \\ &= \langle g \rangle. \end{aligned}$$

$$\text{Ker}(\psi) = \{n \mid g^n = e\}$$

$$g^n = e \Leftrightarrow \text{order}(g) \mid n \quad (\text{or } g \text{ has infinite order}).$$

Suppose g has finite order.

$$\text{Ker}(\psi) = \text{order}(g)\mathbb{Z}.$$

$$\begin{array}{ccc} \text{order}(g)\mathbb{Z} \subseteq \mathbb{Z} & \xrightarrow{\psi} & \langle g \rangle \subseteq G \\ & \searrow & \uparrow \\ & & \mathbb{Z} / \text{order}(g)\mathbb{Z} \end{array}$$

$$\begin{aligned} \langle g \rangle &\cong \mathbb{Z} / \text{order}(g)\mathbb{Z} \\ g^n &\leftarrow n \end{aligned}$$

$\psi:$

Another look at the map $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ when $m|n$.

$$\begin{aligned}\mathbb{Z}/12\mathbb{Z} &\longrightarrow \mathbb{Z}/3\mathbb{Z} \\ a+12\mathbb{Z} &\longrightarrow a+3\mathbb{Z}\end{aligned}$$

$$\begin{aligned}3\mathbb{Z}/12\mathbb{Z} &\subseteq \mathbb{Z}/12\mathbb{Z} \\ &= \ker(\psi)\end{aligned}$$

$$\mathbb{Z}/12\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/3\mathbb{Z}$$

$$\begin{aligned}&\searrow \cong \\ &\frac{(\mathbb{Z}/12\mathbb{Z})}{(3\mathbb{Z}/12\mathbb{Z})}\end{aligned}$$

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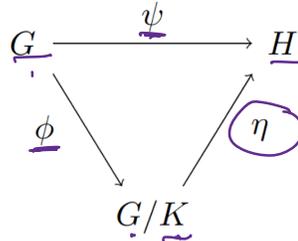


Figure 1: First isomorphism theorem diagram

Want: $\psi(g) = \eta\phi(g)$

Def: $\eta(gk) = \psi(g) \in H$.

Check that this η works.

Ⓐ $\eta(gk) = \eta(g'k) \stackrel{?}{=} gk = g'k$?

$gk = g'k \Leftrightarrow g = g'k \text{ for } k \in K$.

$\eta(gk) = \psi(g) = \psi(g'k) = \psi(g')\psi(k) = \psi(g')$
since $k \in \ker(\psi)$
 $\psi(k) = e$.

Ⓑ $\eta(g_1k) = \eta(g_2k) = \psi(g_1, g_2)$

$\eta(g_1k)\eta(g_2k) = \psi(g_1)\psi(g_2) \stackrel{=}{=} \text{since } \psi \text{ a homomorphism}$

Ⓒ injective. $\eta(g_1k) = \eta(g_2k)$

means $\psi(g_1) = \psi(g_2)$

$\psi(g_1)\psi(g_2)^{-1} = e$

$$\psi(gg_2^{-1}) = e$$

$$g_1g_2^{-1} \in \ker(\psi) = K$$

$$g_1K = g_2K.$$

① If $h \in \psi(A)$, $h = \psi(g)$

for some $g \in A$.

$$\psi(gK) = \psi(g) = h.$$