Homomorphisms: Basics

Definition: Suppose G and H are groups. A homomorphism $\phi : G \to H$ is a function that satisfies the property

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$

for all $g_1, g_2 \in G$. An isomorphism is a homomorphism that is bijective.

A homomorphism is a map that gives a partial relation between the structure of G and H.

Examples

- Let $G = S_n$ and $H = \mathbb{Z}_2$. Define

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is an even permutation} \\ 1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

  Then ϕ is a homomorphism.

Condition: $\varphi(\sigma_1\sigma_2) = \varphi(\sigma_1) + \varphi(\sigma_2)$

σ_1	σ_2	$\sigma_1\sigma_2$	$\varphi(\sigma_1)$	$\varphi(\sigma_2)$	$\varphi(\sigma_1\sigma_2)$
even	even	even	0	0	0
even	odd	odd	0	1	1
odd	even	odd	1	0	1
odd	odd	even	1	1	0

- Let $G = \mathrm{GL}_2(\mathbb{R})$ and let $H = \mathbb{R}^\times$. Then

$$\phi(g) = \det(g)$$

is a homomorphism.

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= ad - bc.$

$\underset{\det}{\phi}(g_1 g_2) = \det(g_1)\det(g_2)$

- Let H be any group and let $h \in H$ be an element. Define $\phi : \mathbb{Z} \to H$ by $\phi(n) = h^n$. Then ϕ is a homomorphism.

$\mathbb{Z} = G$

$H = S_n \quad n \geq 10$

$h = (1\,3\,5\,7\,9)$

$\varphi(n) = h^n$

$\varphi(n) = h^2$

$\varphi(0) = e$

$\varphi(1) = h$

$\varphi(2) = h^2$

$\vdots$

$\varphi(5) = h^5 = e$

$$\boxed{\varphi(n_1+n_2) = h^{n_1+n_2} = h^{n_1} h^{n_2}}$$

- Let $G = \overset{+}{\mathbb{R}}$ and $H = \overset{\cdot}{\mathbb{T}}$, the group of complex numbers of norm 1 with multiplication. Then the map

$$\phi(r) = \text{cis}(r) = \cos(r) + i\sin(r) \qquad (= e^{ir})$$

is a homomorphism.

$$\varphi(r_1 + r_2) = \varphi(r_1)\varphi(r_2)$$

$$\text{cis}(r_1 + r_2) = \text{cis}(r_1)\,\text{cis}(r_2)$$

$$\text{cis}(r_1 + r_2) = \cos(r_1 + r_2) + i\sin(r_1 + r_2) \quad \star$$

$$\text{cis}(r_1)\,\text{cis}(r_2) = (\cos(r_1) + i\sin(r_1))(\cos(r_2) + i\sin(r_2))$$

$$\star\ (\cos(r_1)\cos(r_2) - \sin(r_1)\sin(r_2)) + i(\cos(r_1)\sin(r_2) + \sin(r_1)\cos(r_2))$$

$\star$'s are equal by addition law for sin, cos.