Simple groups

Definition: A group G is simple if its only *normal* subgroups are the trivial group and the entire group G.

Example: The groups $\mathbb{Z}_p$ are simple when p is prime.

Proposition: The only abelian non-trivial simple groups are the groups $\mathbb{Z}_p$ for p prime.

$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7, \ldots$ only non-trivial abelian simple gps.

Proof: In an abelian group, every subgroup is normal. ($gHg^{-1} = H$ is automatic since $gHg^{-1} = gg^{-1}H = H$).
simple + abelian $\Rightarrow$ the only subgroups of G are $\{0\}$ and G. Suppose $G \neq \{0\}$ and choose $x \neq 0$ in G.

Consider $H = \langle x \rangle$. G simple $\Rightarrow$ $H = G$.

So G is cyclic.

- $G = H = \cancel{\mathbb{Z}}$. ($\mathbb{Z}$ is not simple; $3\mathbb{Z} \subseteq \mathbb{Z}$)
  $\mathbb{Z}$ is not simple

- $G = \mathbb{Z}_n$ for some n.
  If n is not prime then choose $d \mid n$, $d > 1$ and $d < n$. Consider $\langle d \rangle \subseteq \mathbb{Z}_n$.
  has n/d elements so it's not all of $\mathbb{Z}_n$. $\mathbb{Z}_n$ is not simple.

- $G = \mathbb{Z}_p$ for p prime.
  $\langle x \rangle \subseteq G$. $\text{order}(x) = p$. $\langle x \rangle = G$.
  G simple.

even permutations in S_n.

The alternating group A_n is simple $n \geq 5$.

For simplicity we will only prove that A_5 is simple. But first notice that:

- A_3 is the group of rotations in S_3, so it's abelian and isomorphic to $\mathbb{Z}_3$, which is simple.

  $A_3 = \{e, (123), (132)\} \leq S_3$

  $\cong \mathbb{Z}_3$ simple and abelian.

- A_4 has 12 elements. The subgroup consisting of

  $$H = \{e, (12)(34), (13)(24), (14)(23)\}$$

  is normal so A_4 is not simple.

Any $g \in A_4. \Rightarrow g \in S_4$

$g(12)(34)g^{-1}$ has same cycle structure: $\underbrace{(ab)(cd)}_{\text{disjoint.}}$

$gHg^{-1} = H$

$geg^{-1} = e$ $\quad g((ab)(cd))g^{-1}$ is another product of disjoint transpositions

Proposition: A_5 is simple. A_n simple for $n \geq 5$.

Proof: See text.

Assume $H \trianglelefteq A_5$.

Use the fact that 3-cycles generate A_5

force 3-cycles into H using conjugation.

induction

$\to A_n$ for

General Problem: Classify all groups.

Classify all finite groups

Start: G finite

If $H \trianglelefteq G$ is normal:

$$\underbrace{H \overset{G}{\longleftrightarrow} G/H}_{\text{smaller.}}$$

If G simple — can't do this.

simple groups are the basic blocks of all groups.