Cosets

The main tool in proving Lagrange's theorem is the idea of a "coset."

Definition: Let H be a subgroup of a group G, and let $g \in G$. Then

$$gH = \{gh : h \in H\} \subseteq G \qquad \left[\, g+H = \{g+h : h \in H\} \,\right]$$

is called a *left coset* of H in G, and

$$Hg = \{hg : h \in H\}$$

is called a *right coset* of H in G.

Examples

$G = \mathbb{Z}$, $H = n\mathbb{Z}$ for some $n \in \mathbb{Z}$. What are the left and right cosets?

$G = \mathbb{Z} \quad H = n\mathbb{Z}$

$n = 2 \quad H = \{\ldots -4, -2, 0, 2, 4, 6, \ldots\}.$

$1 \in G \quad 1 + H = \{1 + h \mid h \in H\}.$

$1 + H = \{-3, -1, 1, 3, 5, 7, \ldots\} = H + 1$

$2 \in G \quad 2 + H = \{-2, 0, 2, 4, 6, 8, 10, \ldots\} = H.$

$a + H = \begin{cases} H & \text{if } a \text{ is even} \\ 1 + H = \text{"odd numbers"} & \text{if } a \text{ is odd} \end{cases}$

$n = 5 \quad H = \{\ldots -20, -15, -10, -5, 0, 5, \ldots\}.$

$1 + H = \{-19, -14, -9, -4, 1, 6, \ldots\} = \{1 + 5k : k \in \mathbb{Z}\}$

$= \{x \in \mathbb{Z} \mid x \equiv 1 \bmod 5\}.$

$r + H = \{r + 5m \mid m \in \mathbb{Z}\} = \{x \in \mathbb{Z} \mid x \equiv r \bmod 5\}.$

$r + H = s + H \iff r \equiv s \bmod 5.$

$H, 1 + H, 2 + H, 3 + H, 4 + H$

$n\mathbb{Z}$ has n cosets

$a + n\mathbb{Z} = b + n\mathbb{Z} \iff a \equiv b \bmod n$

cosets are congruence classes

$G = S_3$ and H is the subgroup of 3-cycles. What are the left cosets?

$H = \{e, (132), (123)\}$ $\quad gH \quad g \in G.$

H

$(12)H$

① $g \in H \quad gH = H.$

② $(12)H = \{(12)e, (12)(132), (12)(123)\}$

$= \{(12), (13), (1)(23) = (23)\}.$

③ $(13)H = \{(13)e, (13)(132), (13)(123)\}$

$= \{(13), (1)(32), (12)\}.$

④ $(23)H = \{(13), (23), (12)\}.$

What are the right cosets?

$H = \{e, (123), (132)\}$

$H(12) = \{e(12), (123)(12), (132)(12)\}$

$= \{(12), (13), (23)\}.$

2 cosets

$H, \quad H(12)$

rotations reflections

$G = S_3$ and H is the subgroup generated by (12). What are the left cosets?

$H = \{e, (12)\}$

gH: $\quad g = (12)$ $\quad (12)H = \{(12)e, (12)(12)\}$
$= \{(12), e\} = H$.

$(23)H = \{(23)e, (23)(12)\}$
$= \{(23), (213)\}$

$(13)H = \{(13)e, (13)(12)\}$
$= \{(13)e, (123)\}$.

$(123)H$
$= \{(123)e, (123)(12)\}$
$= \{(123), (13)\}$

$(132)H$

What are the right cosets?

$H = \{e, (12)\}$

$H(13) = \{(13), (12)(13)\} = \{(13), (132)\}$.

$H(23) = \{(23), (12)(23)\} = \{(23), (231)\}$

Key property of cosets

Theorem: Let G be a group and H a subgroup. Then the left cosets of H in G partition G, in the sense that the following properties hold:

- For $g_1, g_2 \in G$, either $g_1H = g_2H$ or $g_1H \cap g_2H = \emptyset$. In other words, any two cosets are either identical or disjoint.
- G is the disjoint union of the left cosets gH for all $g \in G$.

The same properties hold for the right cosets.

Proof:

1. Assume $g_1H \cap g_2H \neq \emptyset$. Choose $k \in g_1H \cap g_2H$. $k \in G$

2. This means $k = g_1h_1 = g_2h_2$ for $h_1, h_2 \in H$.

$$g_1 = kh_1^{-1} = (g_2h_2)h_1^{-1}$$
$$g_2 = kh_2^{-1} = (g_1h_1)h_2^{-1}$$

3. First we show $g_1H \subset g_2H$.

Let $x \in g_1H$, so $x = g_1h$ for some $h \in H$.

$$x = g_2h_2h_1^{-1}h = g_2(\underbrace{h_2h_1^{-1}h}_{\in H}) \in g_2H$$

4. Then we show $g_2H \subset g_1H$.

Let $x \in g_2H$. Then $x = g_2h$ for some $h \in H$.

$$x = g_1h_1h_2^{-1}h = g_1(\underbrace{h_1h_2^{-1}h}_{\in H}) \in g_1H.$$

So $g_1H = g_2H$

Together this shows the cosets are either disjoint or identical. Since $g \in gH$, every element of g belongs to some coset.

$g \in gH$ $\quad H \ni e$

gH contains ge.

Theorem: The number of left and right cosets of H in G is the same.

Proof: Let L and R be the sets of left and right cosets respectively. Define a map

$$f : L \to R$$

by $f(gH) = Hg^{-1}$.

We will show that f is bijective.

- f is injective

suppose $f(g_1 H) = f(g_2 H)$

$\Rightarrow Hg_1^{-1} = Hg_2^{-1}$

$g_2^{-1} = h g_1^{-1}$ for some h.

$g_2^{-1} g_1 = h$.

$g_1 = g_2 h$ $\quad g_1 \in g_2 H$

$g_1 H \cap g_2 H \neq \emptyset \Rightarrow g_1 H = g_2 H$.

- f is surjective

let Hg be any right coset.

$f(g^{-1}H) = Hg$

so f is surjective.