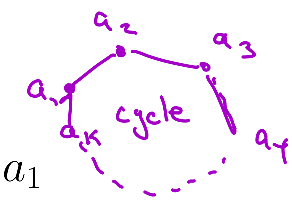
Cycles

Cycles are a more efficient way to work with permutations. A *cycle* σ of length k is a permutation of the form



$$\sigma(a_1) = a_2, \sigma(a_2) = a_3, \ldots, \sigma(a_k) = a_1$$

- We write $(a_1\ a_2\ a_3\ \cdots\ a_k)$ as a shorthand for this cycle.
- If an index i isn't mentioned in a cycle σ, it is fixed, so $\sigma(i) = i$.

$\sigma \in S_5$

$\sigma = (1\,3\,5)$

$\sigma(1) = 3,\ \sigma(3) = 5$

$\sigma(5) = 1 \quad \sigma(2) = 2$

$\sigma(4) = 4$

$$\sigma = \boxed{(135)(42)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

- Cycles are multiplied right to left as with permutations generally

$$\sigma = (13542)$$

$$\tau = (34)$$

$$\sigma\tau = (13542)(34) =$$

$(1\,3\,2)(4\,5)$

Disjoint Cycles Commute

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} = (132)(45) = (45)(132) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

Proposition: Two cycles $\sigma = (a_1 a_2 \cdots a_k)$ and $\tau = (b_1 b_2 \cdots b_r)$ are *disjoint* if $a_i \neq b_j$ for all pairs $1 \leq i \leq k$ and $1 \leq j \leq r$. If σ and τ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

Proof:

$\sigma\tau = (a_1 a_2 \dots a_k)(b_1 \dots b_r)$

$\tau\sigma = (b_1 \dots b_r)(a_1 \dots a_k)$

$\sigma\tau(a_1) = \sigma(a_1) = a_2$

$\tau\sigma(a_1) = \tau(a_2) = a_2$

$\sigma\tau(a_k) = \sigma(a_k) = a_1$

$(i<k)$

$\sigma\tau(a_i) = \sigma(a_i) = a_{i+1}$

$\tau\sigma(a_i) = \tau(a_{i+1}) = a_{i+1}$

$\tau\sigma(a_k) = \tau(a_1) = a_1$

$\sigma\tau(b_i) = \sigma(b_{i+1}) \quad 1 \leq i < r$

$= b_{i+1}$

$\tau\sigma(b_i) = \tau(b_i) = b_{i+1} \quad 1 \leq i < r$

Checking all these cases shows $\sigma\tau = \tau\sigma$.

Products of disjoint cycles

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 5 & 2 \end{pmatrix} = (1\ 6\ 2\ 4)$$

$(1\ 6\ 2\ 4)$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 5 & 6 & 4 \end{pmatrix} = (1\ 3)(4\ 5\ 6)$$

$(1\ 3)(4\ 5\ 6)$

Every permutation is a product of disjoint cycles

Proposition: Any permutation $\sigma \in S_n$ can be written as a product $\sigma = \sigma_1\sigma_2\cdots\sigma_r$ where the σ_i are disjoint cycles.

Proof:

$\sigma \in S_n$

Look at $1, \sigma(1), \sigma^2(1), \dots, \sigma^r(1)$ $\quad \sigma^{r+1}(1) = 1$.

$X_1 \subseteq \{1, \dots, n\}$ $\quad X_1 = \{1, \sigma(1), \dots, \sigma^r(1)\}$ $\quad r$ elements.

Choose $a_2 \in \{1, \dots, n\}$, $a \notin X_1$.

$X_2 = \{a, \sigma(a), \sigma^2(a), \dots, \sigma^{r_2}(a)\}$ $\quad r_2$ elements.

Choose $a_3 \in \{1, \dots, n\}$ $\quad a_3 \notin X_1 \cup X_2$

$X_3 = \{a_3, \sigma(a_3), \dots, \sigma^{r_3}(a_3)\}$, $\quad r_3$ elements.

$\vdots$

Finitely steps.

$X = X_1 \cup X_2 \cdots \cup X_s$ $\quad$ sets are disjoint.

$|S| \leq n$.

$\sigma = (1\ \sigma(1), \dots, \sigma^r(1))(a_2, \sigma(a_2), \dots, \sigma^{r_2}(a_2))(\quad)(\quad)$

ignore X_i if $|X_i| = 1$.

$u \in X$ $\quad u \in X_i$ for some i $\quad u = \sigma^j(a_i)$

$\sigma(u) = \sigma^{j+1}(a_i)$.