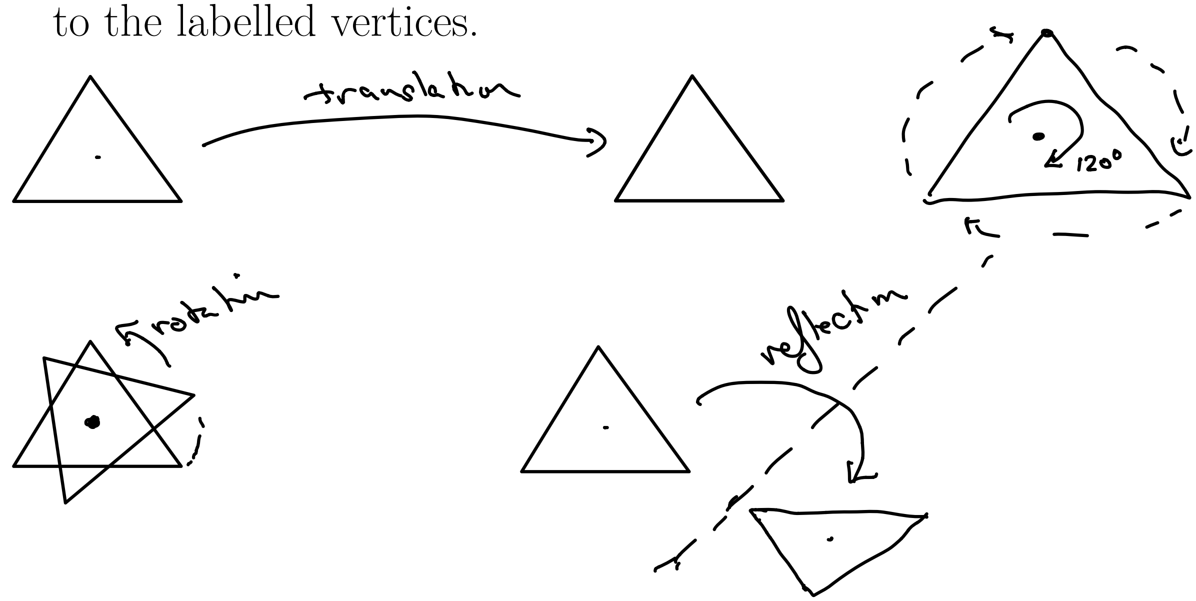
Symmetries of an equilateral triangle

- A *rigid motion* of the Euclidean plane is a transformation that preserves the distances and angles between points. between lines. Rigid motions are combinations of rotations, reflections, and translations.

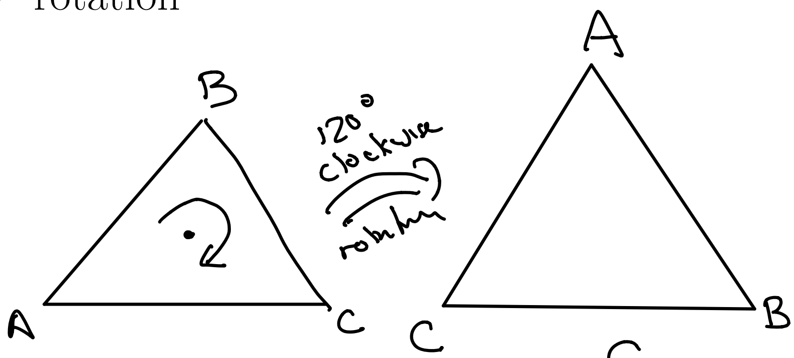
- The (Euclidean) symmetries of a region in the plane are the rigid motions that carry the region back onto itself.

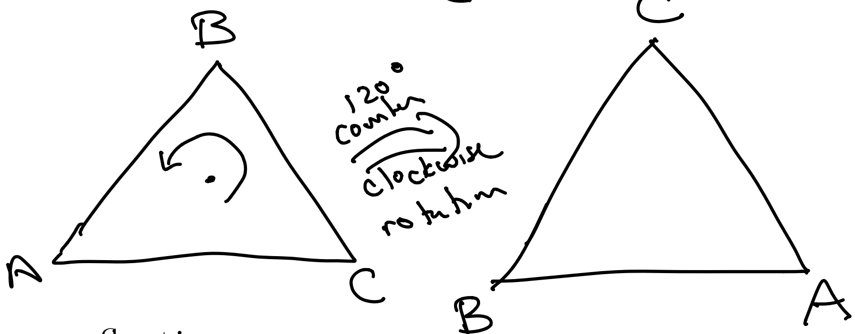
- Thus a symmetry σ of a triangle T is a map $\sigma : T \to T$ that rearranges the edges and vertices according to a rigid motion. We can track the effect of the symmetry by seeing what happens to the labelled vertices.



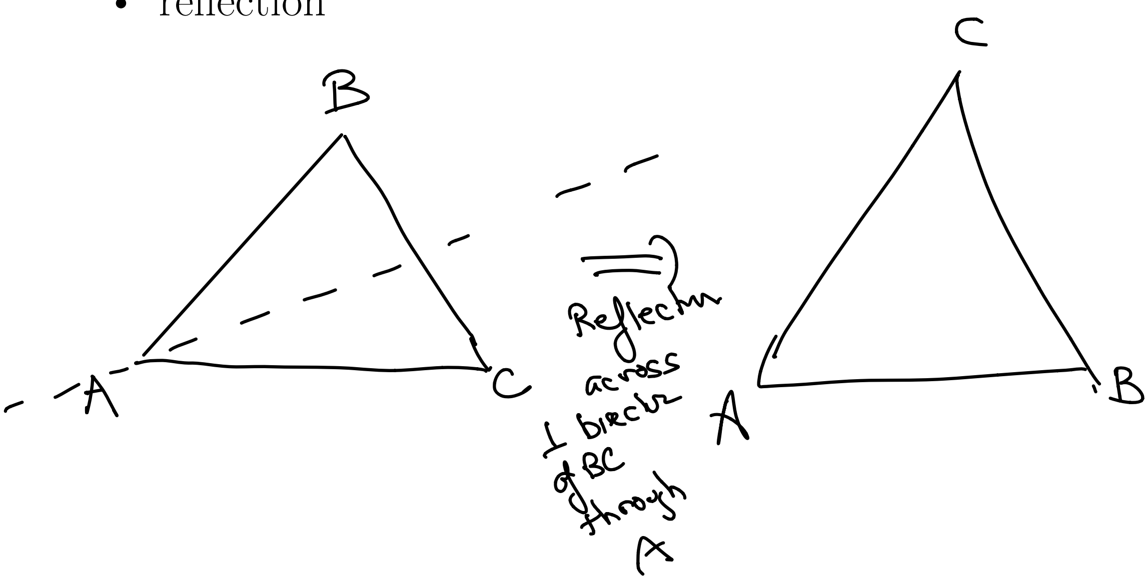
Examples of symmetries of a triangle

- rotation





- reflection



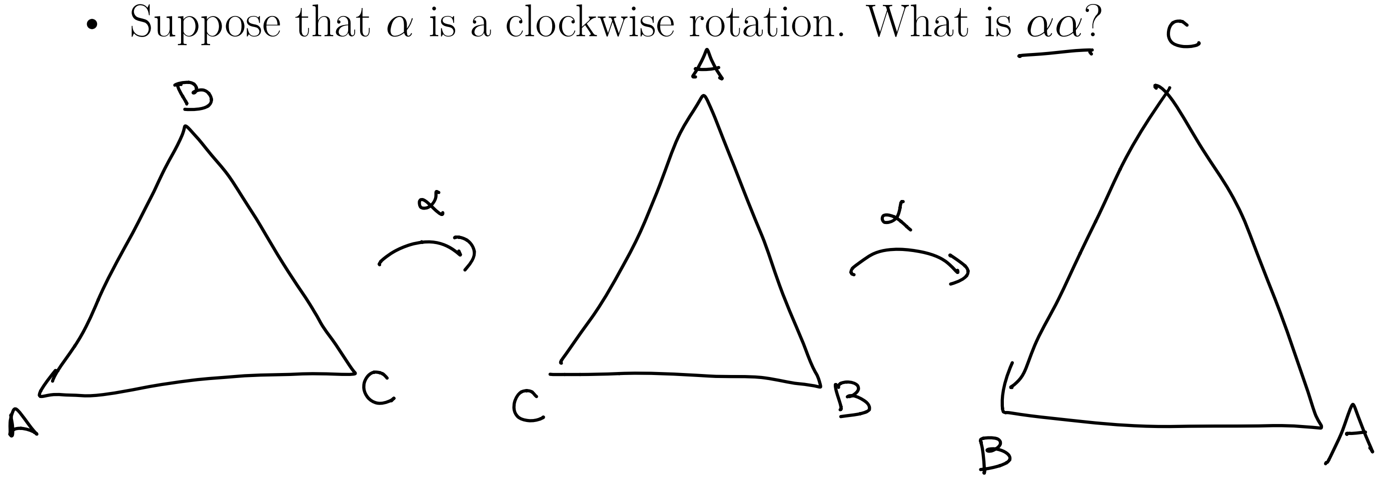
Composition (or "multiplication") of symmetries

Definition: Suppose that α and β are symmetries of an equilateral triangle T. Then the "product" $\alpha\beta$ of α and β is the composition $\alpha \circ \beta : T \to T$, which is another symmetry of the same triangle. *Remember* that $\alpha \circ \beta$ means *first* β, *then* α!.

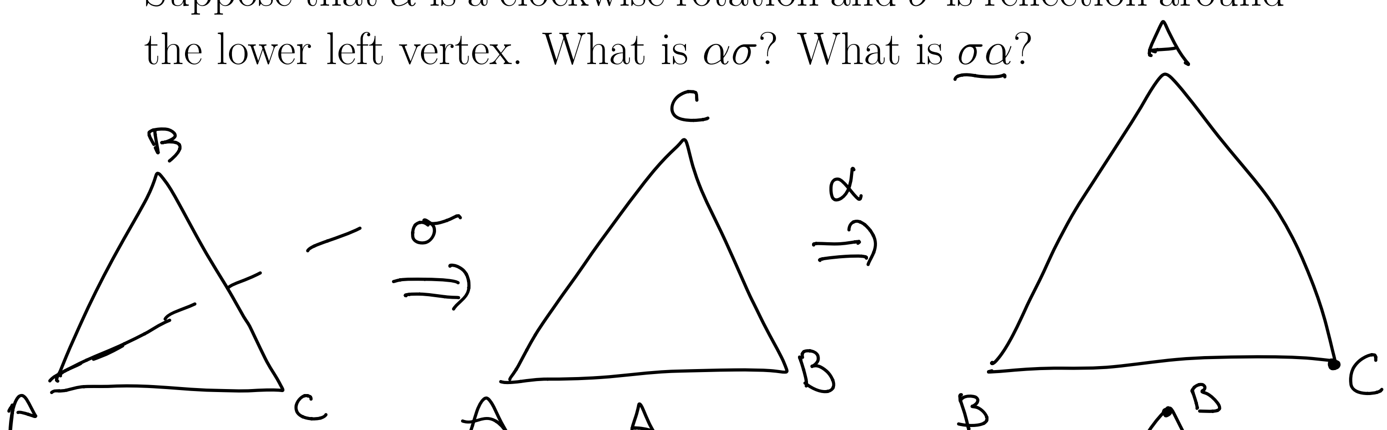
We track the effect of symmetries by watching how the labels on the vertices are affected by them.

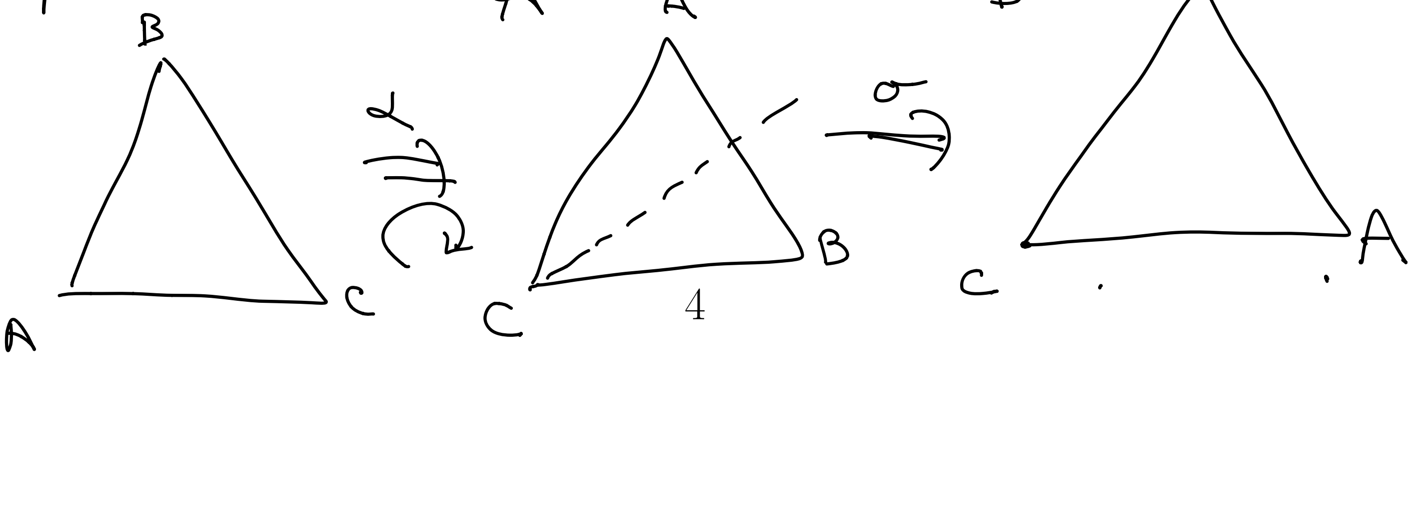
Examples

- Suppose that α is a clockwise rotation. What is $\alpha\alpha$?



- Suppose that α is a clockwise rotation and σ is reflection around the lower left vertex. What is $\alpha\sigma$? What is $\sigma\alpha$?





The set of symmetries

Proposition: There are six symmetries of an equilateral triangle.

Any symmetry rearranges the vertices A, B, C.

Only six ways to rearrange A, B, C.

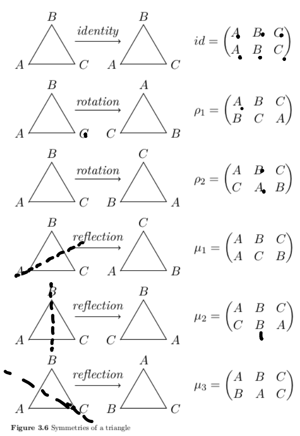
2 choices

$3 \cdot 2 \cdot 1 = 6.$

1 choice

3 choices

6 symmetries:
identity
2 rotations
3 reflections



ρ_1

ρ_2

Figure 1: Chapter 3, Figure 6

The multiplication table for symmetries of a triangle

$\circ$	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
id	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	id	μ_3	μ_1	μ_2
ρ_2	ρ_2	id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

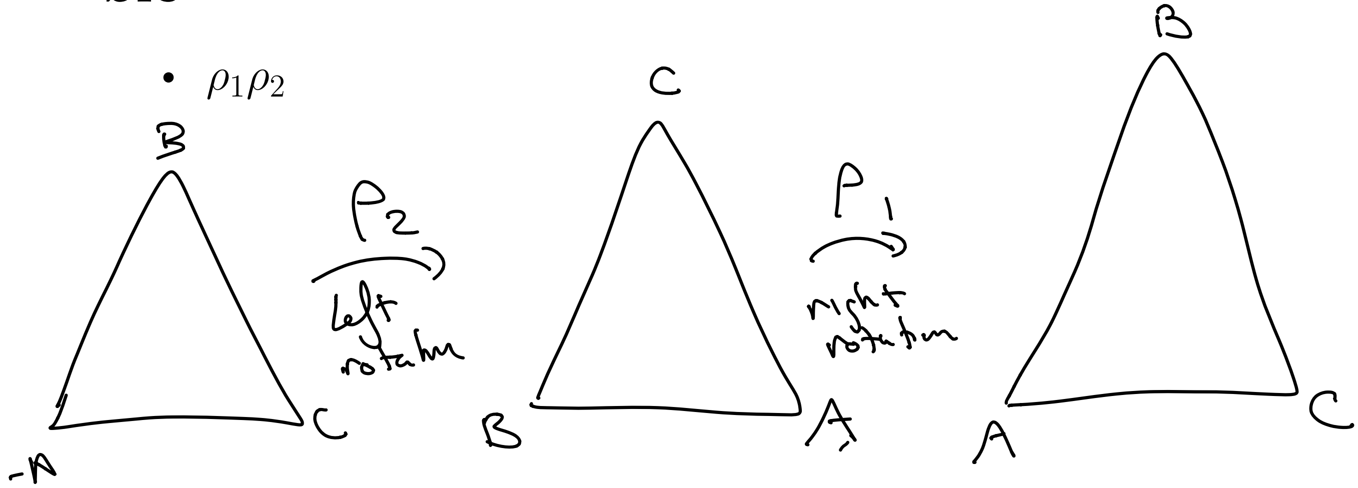
$\rho_2 \mu_1 = \mu_2$

Figure 3.7 Symmetries of an equilateral triangle

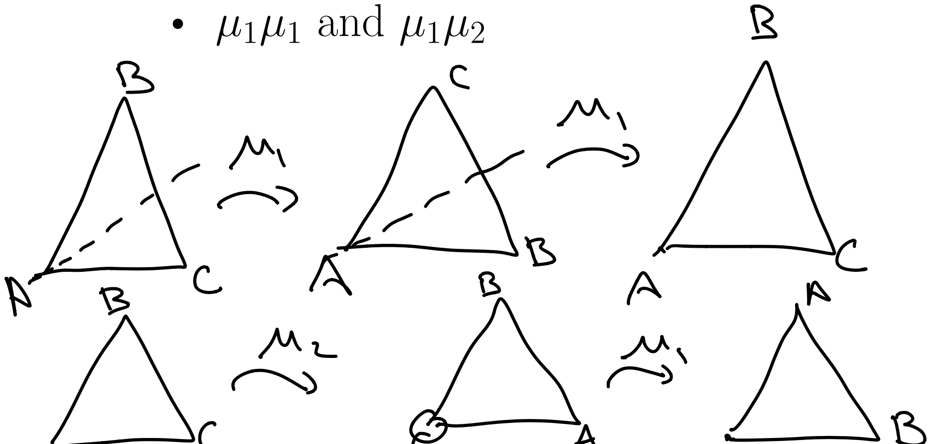
Figure 2: Chapter 3, Figure 7

Checking some entries of the multiplication table

- $\rho_1\rho_2$



- $\mu_1\mu_1$ and $\mu_1\mu_2$

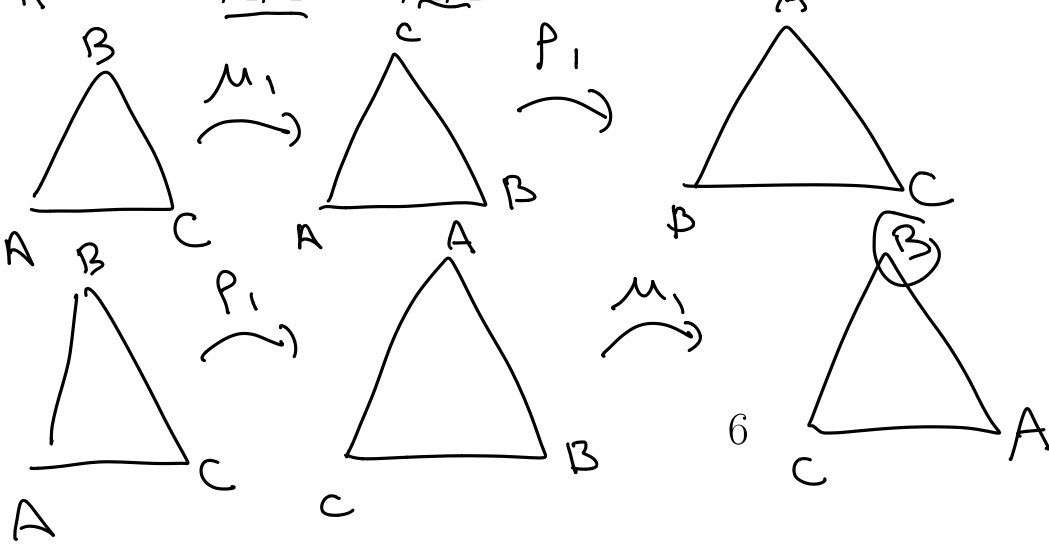


$\mu_i\mu_i = \text{identity}$

Right rotation

$\mu_1\mu_2 = \rho_1$

- $\rho_1\mu_1$ and $\mu_1\rho_1$

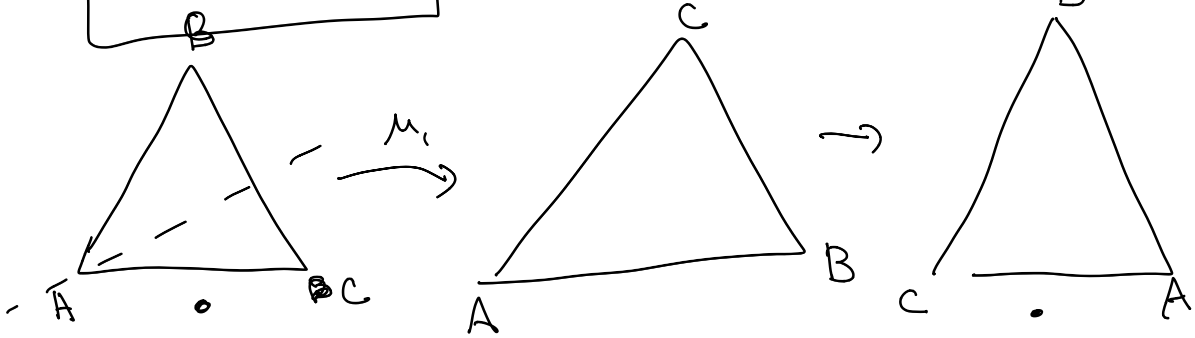


$\rho_1\mu_1 = \mu_3$

$\mu_1\rho_1 = \mu_2$

$$\boxed{\rho_2 \mu_1 = \mu_2}$$

ρ_2 = left rotation
μ_1 = flip around A



$$\begin{pmatrix} B & A & C \\ B & C & A \end{pmatrix}$$