$I_n = \{1, 2, 3, \ldots, n\}$

$S_n = \{ f : I_n \to I_n \mid f \text{ is bijective} \}$

operation is composition of functions

$\underline{n=5}$ $\quad f : I_5 \to I_5$

$$\begin{array}{c|ccccc} i & 1 & 2 & 3 & 4 & 5 \\ f(i) & 3 & 1 & 2 & 5 & 4 \end{array}$$

$f(1)=3, f(2)=1; f(3)=2; f(4)=5, f(5)=4.$

$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$. $\quad \# S_n = n!$ $\quad S_n = \{\text{permutations of } n \text{ elements}\}$.

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$

$fg \leftrightarrow f \circ g$ $\quad fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$

$(f \circ g)(1) = \underline{f(g(1))} = f(2) = 1$

$(f \circ g)(2) = f(g(2)) = f(1) = 3$

$(f \circ g)(3) = f(g(3)) = f(5) = 4$

$(f \circ g)(4) = f(g(4)) = f(3) = 2$

$(f \circ g)(5) = f(g(5)) = f(4) = 5$

• $(f \circ g) \circ h \overset{?}{=} f \circ (g \circ h)$ True because composition is associative

•) identity $\quad f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ $\quad (n=5)$ $\quad f = \text{identity}$ $\quad f(i) = i$

$g \in S_n$ $\quad (g \circ f)(i) = g(f(i)) = g(i)$ for all i.

therefore $g \circ f = g$

$(f \circ g)(i) = f(g(i)) = g(i)$ $\quad f \circ g = g$

•) $\underline{\text{inverse}}$: Thm: If $f : I_n \to I_n$ bijective, then f has an inverse.

More detail: Given $f: I_n \to I_n$.

~~Let~~ Take any $j \in I_n$. We know there is exactly one i so that $f(i) = j$.

Then $f^{-1}(j) = i$.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}. \qquad f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

$f(2) = 1$

$f^{-1}(1) = 2$

$f(3) = 2$

$f^{-1}(2) = 3$

$(f \circ f^{-1})(1) = f(2) = 1$

$(f \circ f^{-1})(2) = f(f^{-1}(2)) = f(3) = 2$

$\vdots$

$(f \circ f^{-1})(i) = i$ for $i = 1, \ldots, 5$.

$$\begin{pmatrix} 3 & 1 & 2 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

S_n is not abelian except when $n = 2$.

$n = 2$: $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$S_3, \ldots$

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 3 & 4 & \cdots & n \end{pmatrix}$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 3 & 2 & 4 & \cdots & n \end{pmatrix}$$

$$ab = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 3 & 1 & 4 & 5 & \cdots & n \end{pmatrix}$$

$$ba = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 1 & 2 & 4 & \cdots & n \end{pmatrix}$$