Set equality

The assertion that two sets A and B are equal is equivalent to saying that

$$x \in A \Leftrightarrow x \in B.$$

In other words, x is in A if and only if x is in B. Now $(x \in A) \Leftrightarrow (x \in B)$ is the same as

$$\underbrace{[(x \in A) \implies (x \in B)]}_{A \subseteq B} \text{ AND } \underbrace{[(x \in B) \implies (x \in A)]}_{B \subseteq A}$$

and this is just $A \subseteq B$ and $B \subseteq A$.

So we prove two sets are equal by proving BOTH $A \subseteq B$ and $B \subseteq A$.

$A = B \iff (A \subseteq B$ and $B \subseteq A)$

Euclid's algorithm

Here's what we proved in the discussion in Chapter 7.

Proposition: Let $d = \gcd(a, b)$ and let m be any integer. Then there exist k and l such that $m = ak + bl$ if and only if $d|m$.

Set version:

$a=7, b=3$
$d=1$

Proposition: Let a and b be natural numbers, and let $d = \gcd(a, b)$. Define sets $A = \{dn : n \in \mathbb{Z}\}$ and $B = \{ax + by : x, y \in \mathbb{Z}\}$. Then $A = B$.

$\mathbb{Z} = \{7x+3y \mid x,y \in \mathbb{Z}\}$
$\parallel$
$\mathbb{Z} = \{1 \cdot n \mid n \in \mathbb{Z}\}$

Here:

- $A \subseteq B$ means that every multiple of d can be written in the form $ax + by$.
- $B \subseteq A$ means that every number of the form $ax + by$ is a multiple of d.

More examples

Proposition: Let a and b be prime numbers. Let $A = \{da : d \in \mathbb{Z}\}$ and $B = \{db : d \in \mathbb{Z}\}$. Then $A \cap B = \{dab : d \in \mathbb{Z}\}$.

a, b are prime:

$A = \{da : d \in \mathbb{Z}\}$ $B = \{db : d \in \mathbb{Z}\}$.

$A \cap B = \{abd : d \in \mathbb{Z}\}$

$a = 3$ $b = 5$

$A = \{\ldots, 6, -3, 0, 3, 6, \ldots\}$ 15

$B = \{-15, -10, -5, 0, 5, 10, 15, \ldots\}$

$A \cap B = \{dab : d \in \mathbb{Z}\}$. If $x \in A$ and $x \in B$ then $x \in \{dab : d \in \mathbb{Z}\}$.

(1) Show $A \cap B \subseteq \{dab : d \in \mathbb{Z}\}$.

$x \in A$ means $x = d_1 a$. $x \in B = d_2 b$. Both a, b appear in prime factors of x. Therefore ab is a divisor of x.

so $x = abd$. which means $A \cap B \subseteq \{dab : d \in \mathbb{Z}\}$

More examples

Proposition: If A, B, and C are sets then
$A \times (B \cap C) = (A \times B) \cap (A \times C)$. (this is problem 17 on page 171)

Proof: First we show $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

In other words we show $A \times (B \cap C) \subseteq A \times B$ and $A \times C$.

Let $(a, x) \in A \times (B \cap C)$

so $a \in A$ and $x \in B \cap C$.

Therefore $x \in B$ and $x\, C$

so $(a, x) \in A \times B$ and $(a, x) \in A \times C$. This shows

$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

Now we show $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Let $(a, x) \in (A \times B) \cap (A \times C)$.

So $(a, x) \in A \times B$ so $x \in B$

and $(a, x) \in A \times C$ so $x \in C$

Therefore $x \in B \cap C$ and $(a, x) \in A \times (B \cap C)$.

This shows $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

More examples

Proposition: Prove that $\{12a + 4b : a, b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$.

$\{12a+4b : a,b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$

1. $\{12a+4b : a,b \in \mathbb{Z}\} \subseteq \{4c : c \in \mathbb{Z}\}$.

$\Leftarrow$ if $y = 12a+4b$, then $y = 4c$ for some $c \in \mathbb{Z}$.

$y = 12a+4b = 4(3a+b)$ so let $c = 3a+b$

and we see $y = 4c$

so $\{12a+4b : a,b \in \mathbb{Z}\} \subseteq \{4c : c \in \mathbb{Z}\}$.

2. $\{4c\} \subseteq \{12a+4b : a,b \in \mathbb{Z}\}$,

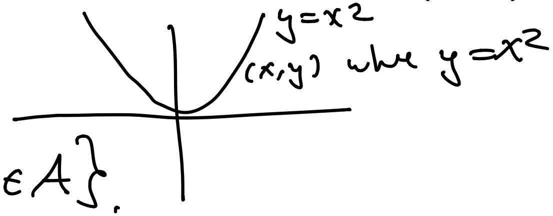
if $x = 4c$ then $x = 12a+4b$ for $a,b \in \mathbb{Z}$,

Set $a = 0$, $b = c$: $x = 4c$.

$\{4c\} \subseteq \{12a+4b\}$ so sets are equal.

More examples

Proposition: Let $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$. Let B be the set of real numbers z such that there exists $x \in \mathbb{R}$ such that $(x, z) \in A$. Then $B = \{z \in \mathbb{R} : z \geq 0\}$.



$A = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$

$B = \{z : \exists\, x \in \mathbb{R} \text{ with } (x,z) \in A\}$.

$B = \{z \in \mathbb{R} : z \geq 0\}$.

Is $3 \in B$? is there an ~~x~~ $x \in \mathbb{R}$ so that $(x,3) \in A$.
$(x,3) \in A$ means $3 = x^2$. Answer yes: $3 \in B$ because
$(\sqrt{3}, 3) \in A$.

Is $-1 \in B$? is there an $x \in \mathbb{R}$ so that $(x,-1) \in A$?
No: because $-1 = x^2$ has no solutions.

$B = \{z \in \mathbb{R} \mid z = x^2\}$ has a solution in $\mathbb{R}$, true

$B \subseteq \{z \in \mathbb{R} : z \geq 0\}$. if $z < 0$ then $z \notin B$ because $x^2 = z$ has no solution.

~~A $z \in \mathbb{R}$~~ $\{z \in \mathbb{R} \mid z \geq 0\} \subseteq B$: if $z \geq 0$, then $x^2 = z$ has a solution in $\mathbb{R}$, which is true.

$\{dab : d \in \mathbb{Z}\} \subseteq A \cap B$

Show: $x = dab$ for some $d \Rightarrow x \in A$ and $x \in B$

if $x = dab$ for some d then $x = d_1 a$ and $x = d_2 b$

But $x = (db)a$ and $x = (da)b$

so $d_1 = db$ and $d_2 = da$ we show

that $x \in A \cap B$

$\{dab\}_{d \in \mathbb{Z}} = A \cap B.$

$A \times (B \cap C) = A \times B \cap A \times C$