Existence Proofs

Review of universal quantifiers

A theorem asserting the truth of a conditional statement is typically a "for all" statement.

Theorem: If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, it is continuous.

Here there is an implicit *universal quantifier.*

Theorem: For all functions $f : \mathbb{R} \to \mathbb{R}$, f differentiable implies f continuous.

Another example

Theorem: An $n \times n$ matrix A with real entries is invertible if and only if $det(A) \neq 0$.

This is asserting that:

For all $n \times n$ matrices A with real entries, A is invertible if and only if $det(A) \neq 0$.

$\forall A$, A real $n \times n$ matrix, A invertible $\iff det(A) \neq 0$

Existence claims

Some theorems assert the existence of an object with particular properties.

Proof of an existence theorem requires you to present an example.

Definition: A Pythagorean Triple is an element (a, b, c) of $\mathbb{Z}^3$ such that

$$c^2 = a^2 + b^2.$$

Theorem: A Pythagorean triple exists.

Proof: Let $a = 3$, $b = 4$, and $c = 5$. Then $c^2 = 25 = a^2 + b^2$.

Existence claims can be hard to establish

Theorem: There exist integers A, B, and C so that

$$A/(B+C)+B/(A+C)+C/(A+B)=4.$$

Proof: Let

A= 154476802108746166441951315019919837485664325669565431700026634898253202035277999
B= 36875131794129999827197811565225474825492979968971970996283137471637224634055579
C= 4373612677928697257861252602371390152816537558161613618621437993378423467772036

Figure 1: Big Numbers

Then these values satisfy the given equation. (Check this if you can!)

- verification requires work
- no clue given as to how to find this; and, in fact, it's hard.

Millenium Problems